Solutions to Exam for Random Processes, 5 points/7.5 ECTS

August 24, 2005, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Sheet of formulae attached to the exam, calculator and Mathematics Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Assume that $\{X_t : t \in \mathbb{Z}\}$ is an $AR(2)$ process with parameters a_1 and a_2. Show that its spectral density is $R_X(f) = 1/(1 + a_1 e^{-i2\pi f} + a_2 e^{-i4\pi f})$. (4p)

   Solution: (See the compendium, pages 88–89, and apply to the special case when $p = 2$.) □

2. Let $\{X_t : t \in \mathbb{R}\}$ be a Poisson process with intensity 1, let $Y_t = e^{3X_t}$ and let $Z_t = Y_t/Y_{t-2}$ for all $t \in \mathbb{R}$. Determine the
   (a) probability function of the variable Z_t. (3p)
   (b) expectation function of the process $\{Z_t\}$. (4p)

   Solution:

   (a)
$$\begin{aligned} P(Z_t = z) &= P\Big(\frac{Y_t}{Y_{t-2}} = z\Big) \\ &= P\Big(\frac{e^{3X_t}}{e^{3X_{t-2}}} = z\Big) \\ &= P(e^{3(X_t - X_{t-2})} = z) \\ &= P(3(X_t - X_{t-2}) = \ln z) \\ &= P(X_t - X_{t-2} = \tfrac{1}{3}\ln z) \end{aligned}$$

   $X_t - X_{t-2} \in Poi(t-(t-2)) = Poi(2) \;\Rightarrow\; P(Z_t = z) = \dfrac{2^{\frac{1}{3}\ln z}}{(\frac{1}{3}\ln z)!}\, e^{-2}$ where $z \in \{e^{3k} : k \in \mathbb{N}\}$.

   (b)

$$
\begin{aligned}
E(Z_t) &= E(e^{3(X_t - X_{t-2})}) \\
&= \sum_{k=0}^{\infty} e^{3k} P(X_t - X_{-2} = k) \\
&= \sum_{k=0}^{\infty} e^{3k} \frac{2^k}{k!} e^{-2} \\
&= \sum_{k=0}^{\infty} \frac{(2e^3)^k}{k!} e^{-2e^3} e^{2e^3-2} \\
&= e^{2e^3-2} \sum_{k=0}^{\infty} \frac{(2e^3)^k}{k!} e^{-2e^3} \\
&= e^{2e^3-2}
\end{aligned}
$$

□

3. Let the random process $\{X_t : t \in \mathbb{R}\}$ in continuous time have cvf $r_X(\tau) = 2/(1 + 4\pi^2\tau^2)$ and let $Y_k = X_{2k}$ where $k = 0, \pm 1, \pm 2, \ldots$. (Thus the sequence $\{X_t\}$ is sampled at times $t = \ldots, -4, -2, 0, 2, 4, \ldots$) Calculate the spectral density, $R_Y(f)$, of the sampled process $\{Y_t\}$. (4p)

Solution: $r_X(\tau) = \frac{2}{1+4\pi^2\tau^2}$
$\Rightarrow$ $R_X(f) = e^{-|f|}$ (according to table)
$\Rightarrow$ $R_Y(f) = \frac{1}{d}\sum_{k=-\infty}^{\infty} R_X(\frac{f+k}{d}) = \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{\frac{1}{2}|f+k|} =$
$= \frac{1}{2}\Big(\sum_{k=-\infty}^{-1} e^{\frac{1}{2}(f+k)} + e^{-\frac{1}{2}|f|} + \sum_{k=1}^{\infty} e^{-\frac{1}{2}(f+k)} \Big)$
$= \frac{1}{2}\Big(e^{f/2}\sum_{k=1}^{\infty} e^{-k/2} + e^{-|f|/2} + e^{-f/2}\sum_{k=1}^{\infty} e^{-k/2} \Big)$
$= \frac{1}{2}\Big((e^{f/2} + e^{-f/2}) \dfrac{e^{-1/2}}{1 - e^{-1/2}} + e^{-|f|/2} \Big) = \dfrac{\cosh \frac{f}{2}}{\sqrt{e} - 1} + e^{-|f|/2}$ □

4. Let $\{W_t\}$ be a standard Wiener process and define $X_t = e^{-t/2} W_{e^t}$.
   (a) Determine the distribution of X_t. (4p)
   (b) Is the process $\{X_t\}$ strongly stationary? (3p)

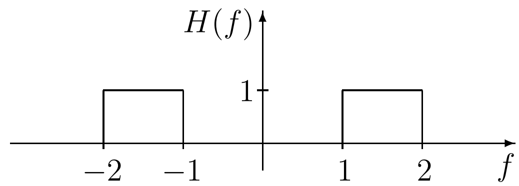
Solution:

(a) Let us first calculate the density function of X_t:
$\frac{d}{dx}P(X_t \leq x) = \frac{d}{dx}P(e^{-t/2}W_{e^t} \leq x) = \frac{d}{dx}P(\underbrace{W_{e^t}}_{\in N(0,\sqrt{e^t})} \leq xe^{t/2}) = F'_{W_t}(xe^{t/2})\,\frac{d}{dx}\{xe^{t/2}\} =$
$e^{t/2}\,\frac{1}{\sqrt{2\pi e^t}}\,e^{-\frac{1}{2e^t}(xe^{t/2})^2} = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}$ which we recognise as the probability density function of a standard normally distributed random variable, i.e.

$X_t \in N(0,1)$.

(b) For $\tau > 0$ we have that $r_W(\tau) = C(X_t, X_{t+\tau}) = C(e^{-t/2}W_{e^t}, e^{-(t+\tau)/2}W_{e^{t+\tau}}) = e^{-t/2}e^{-t/2-\tau/2}C(W_{e^t}, W_{e^{t+\tau}}) = e^{-t}e^{-\tau/2}\min(e^t, e^{t+\tau}) = e^{-t}e^{-\tau/2}e^t = e^{-\tau/2}$. Thus for $\tau \in \mathbb{R}$, $r_W(\tau) = e^{-|\tau|/2}$ which means that $\{X_t\}$ is weakly stationary. Further $\{W_t\}$ is a Wiener process, i.e. a Gaussian process, meaning that $\{X_t\}$ is Gaussian. But a weakly stationary Gaussian process is strongly stationary, so $\{X_t\}$ is! □

5. A weakly stationary process $\{X_t\}$ with expectation 1 and spectral density function R_X is filtered in an ideal bandpass filter with transfer function $H(f)$ as plotted below.



Express the mean effect (i.e. $E(Y_t^2)$) of $\{Y_t\}$ in terms of $R_X(f)$. (4p)

Solution: $V(Y_t) = E(Y_t^2) - E(Y_t)^2 \Rightarrow E(Y_t^2) = V(Y_t) + E(Y_t)^2$ so we need to calculate $E(Y_t)$ and $V(Y_t)$.
We have that $h(\tau) = \int e^{i2\pi f\tau}H(f)\,df = \ldots = \frac{\sin(4\pi\tau)-\sin(2\pi\tau)}{\pi\tau}$ and $E(X_t) = 1 \Rightarrow$ $E(Y_t) = E(\int h(s)X_{s-t}\,ds) = \int h(s)E(X_{s-t})\,ds = \int h(s)\,ds = \int \frac{\sin(4\pi s)-\sin(2\pi s)}{\pi s}\,ds = \int \frac{\sin(2\pi s)(\cos(2\pi s)-1)}{\pi s}\,ds = 0$.
According to Parseval's formula we have that $R_Y = |H|^2 R_X$. Therefore $E(Y_t^2) = V(Y_t) = \int R_Y(f)\,df = \int |H|^2 R_X(f)\,df = 2\int_1^2 R_X(f)\,df$ □

6. A stationary Gaussian process $\{X_t : t \in \mathbb{R}\}$ has expectation 0 and cvf $r_X(\tau) = \frac{1}{\pi}e^{-\pi\tau^2/2}$. Determine the crosscovariance function $r_{X,X'}(\tau) = C(X_t, X'_{t+\tau})$. (4p)

Solution:
$r_X(\tau) = \frac{1}{\pi}e^{-\pi\tau^2/2}$. $r_{X,X'}(\tau) = r'_X(\tau) = \frac{1}{\pi}(-2\pi\tau/2)e^{-\pi\tau^2/2} = -\tau e^{-\pi\tau^2/2}$. □