Solutions to Exam for Random Processes, 5 points/7.5 ECTS

December 17, 2005, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that the cvf of an $AR(p)$ process satisfies the Yule-Walker equations. (3p)

   Solution: (See the compendium, p 89.) □

2. Let $\{N_t : t \in \mathbb{R}^+\}$ be a Poisson process with parameter 2. Calculate
   (a) $V(N_2 + N_3)$. (3p)
   (b) $E((-1)^{N_t})$. (4p)

   Solution:
   (a) $V(N_2 + N_3) = V(N_2) + V(N_3) + 2C(N_2, N_3) = 2 + 2 + 2\min(2, 3) = 8$.
   (b) $E((-1)^{N_t}) = \sum_{k=0}^{\infty}(-1)^k P(N_t = k) = \sum_{k=0}^{\infty}(-1)^k \frac{(2t)^k}{k!} e^{-2t} =$
   $= e^{-2t} \sum_{k=0}^{\infty} \frac{(-2t)^k}{k!} = e^{-2t}e^{-2t} = e^{-4t}$. □

3. Let $\{X_t : t \in \mathbb{R}\}$ be a weakly stationary process with spectral density function $R_X(f) = \delta_{-1}(f) + \delta_1(f)$. Calculate $C(X_t, X_{t+0.5})$. (3p)

   Solution: $R_X(f) = \delta_{-1}(f) + \delta_1(f) \;\Rightarrow\; r_X(\tau) = \int_{\mathbb{R}}(\delta_{-1}(f) + \delta_1(f))e^{i2\pi f\tau}df =$
   $= e^{-i2\pi\tau} + e^{i2\pi\tau} = 2\cos(2\pi\tau) \;\Rightarrow\; C(X_t, X_{t+0.5}) = r_X(0.5) = 2\cos\pi = -2$. □

4. Determine the cvf, $r_X(\tau)$, of an $MA(3)$ process with coefficients c_0, c_1, c_2 and c_3 where $c_0 = 2c_1 = 4c_2 = 8c_3 = 1$ and with $\sigma_\epsilon^2 = 64$. (4p)

Solution: $c_0 = 1$, $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{4}$, $c_3 = \frac{1}{8}$ so

$$\begin{aligned} r_X(\tau) &= \begin{cases} \sigma_\epsilon^2 \sum_{j-k=\tau} c_j c_k & \text{if } |\tau| \le 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 64(1^2 + (\frac{1}{2})^2 + (\frac{1}{4})^2 + (\frac{1}{8})^2) & \text{if } \tau = 0 \\ 64(1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{8}) & \text{if } |\tau| = 1 \\ 64(1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{8}) & \text{if } |\tau| = 2 \\ 64(1 \cdot \frac{1}{8}) & \text{if } |\tau| = 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 85 & \text{if } \tau = 0 \\ 42 & \text{if } |\tau| = 1 \\ 20 & \text{if } |\tau| = 2 \\ 8 & \text{if } |\tau| = 3 \\ 0 & \text{if } |\tau| \ge 4 \end{cases} \end{aligned}$$

□

5. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process with ef $m_X = 0$ and cvf $r_X(\tau) = 2e^{-|\tau|}$.
   (a) Calculate $P(X_t > 2)$. (3p)
   (b) Calculate $P(\int_{-1}^{1} X_t\, dt > 2)$. (3p)

Solution:

(a) $P(X_t > 2) = 1 - P(X_t \le 2) = 1 - \Phi(\frac{2-0}{\sqrt{r_X(0)}}) = 1 - \Phi(\sqrt{2}) = 0.0792.$

(b) $\{X_t\}$ Gaussian $\Rightarrow$ $\int_{-1}^{1} X_t\, dt$ normally distributed with

$\mu = E\Big(\int_{-1}^{1} X_t\, dt\Big) = \int_{-1}^{1} m_X\, dt = 0$ and

$$\sigma^2 = V\Big(\int_{-1}^{1} X_t\, dt\Big) = C\Big(\int_{-1}^{1} X_s\, ds, \int_{-1}^{1} X_t\, dt\Big) = \int_{-1}^{1}\int_{-1}^{1} r_X(s-t)\, ds\, dt =$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2e^{-|s-t|} ds\, dt = 2\int_{-1}^{1}\Big(\int_{-1}^{1} e^{-|s-t|} ds\Big) dt =$$

$$= \left\{ \begin{matrix} u = s-t & s=-1 \Leftrightarrow u = -1-t \\ du = ds & s = 1 \Leftrightarrow u = 1-t \end{matrix} \right\} 2\int_{-1}^{1}\Big(\int_{-1-t}^{1-t} e^{-|u|} du\Big) dt =$$

$$= 2\int_{-1}^{1}\Big(\int_{-1-t}^{0} e^{u} du + \int_{0}^{1-t} e^{-|u|} du\Big) dt = 2\int_{-1}^{1} (e^0 - e^{-1-t} + (-e^{-(1-t)}) - (-e^{-0}))\, dt =$$

$$= 2\int_{-1}^{1} (2 - e^{-1}(e^{-t} + e^{t}))\, dt = 2(2(1-(-1)) - e^{-1}(-e^{-1} + e^1 + e^1 - e^{-1})) =$$

$$= 4(1 + e^{-1}) = 5.47 \;\Rightarrow\; P(\textstyle\int_{-1}^{1} X_t\, dt > 2) = 1 - \Phi(\frac{2-0}{\sqrt{5.47}}) = 0.1977.$$ □

6. Let $\{X_t : t \in \mathbb{R}\}$ be shot noise wiht intensity $\lambda = 30$ and impulse function $g(t) = \begin{cases} x^2 & \text{when } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$. Determine the cvf of $\{X_t\}$. (3p)

Solution: $r_X(\tau) = \lambda \int g(u)g(u-\tau)\, du$

In this problem $g(u) = \begin{cases} u^2 & \text{when } 0 < u < 1 \\ 0 & \text{o.w.} \end{cases}$

$\Rightarrow\ g(u-\tau) = \begin{cases} (u-\tau)^2 & \text{when } 0 < u-\tau < 1 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} (u-\tau)^2 & \text{when } \tau < u < 1+\tau \\ 0 & \text{o.w.} \end{cases}$

so $g(u)g(u-\tau) \neq 0$ when both $g(u) \neq 0$ and $g(u-\tau) \neq 0$, i.e. when $\tau \le u \le 1$ if $0 \le \tau \le 1$ and when $0 \le u \le 1+\tau$ if $-1 \le \tau \le 0$.

If $0 \le \tau \le 1$, then $r_X(\tau) = 30\int_\tau^1 u^2(u-\tau)^2 du = 30\int_\tau^1 u^2(u^2 - 2u\tau + \tau^2)\, du = [\frac{u^5}{5} - 2\tau\frac{u^4}{4} + \tau^2\frac{u^3}{3}]_\tau^1 = 6 - 15\tau + 10\tau^2 - \tau^5$.

If $-1 \le \tau \le 0$, then $r_X(\tau) = 30\int_0^{1+\tau} u^2(u-\tau)^2 du =$
$= 30(\frac{(1+\tau)^5}{5} - 2\tau\frac{(1+\tau)^4}{4} + \tau^2\frac{(1+\tau)^3}{3}) = 6 + 15\tau + 10\tau^2 + \tau^5$, i.e. totally

$r(\tau) = \begin{cases} 6 - 15|\tau| + 10\tau^2 - |\tau|^5 & \text{if } |\tau| < 1 \\ 0 & \text{otherwise} \end{cases}$ □

7. Let $\{X_t\}$ be a weakly stationary process with cvf $r_X(\tau) = e^{-\tau^2}$. Determine approximately how many observations $X_1, X_2, \ldots, X_n$ are needed for the variance of the expectation estimator $m_n^* = \frac{1}{n}\sum_{t=1}^n X_t$ to be smaller than 0.1. (4p)

Solution: Apparently $\sum_{\tau\in\mathbb{Z}} |r_X(\tau)| = \sum_{\tau\in\mathbb{Z}} e^{-\tau^2} < \infty$ so $nV(m_n^*) \approx \sum_{\tau\in\mathbb{Z}} e^{-\tau^2}$ for large n (according to the summary of formulae), where

$\sum_{\tau\in\mathbb{Z}} e^{-\tau^2} = e^0 + 2e^{-1} + 2\sum_{\tau=2}^\infty e^{-\tau^2} \approx 1 + 2e^{-1} + 2\int_{1.5}^\infty e^{-t^2} dt$.

Since the normal density function with $\mu = 0$ is $\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$ we have with $\sigma^2 = \frac{1}{2}$ that $\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Thus with $Y \in N(0, \frac{1}{2})$ we have that $\int_{1.5}^\infty e^{-t^2} dt = \sqrt{\pi}\int_{1.5}^\infty \frac{1}{\sqrt{\pi}} e^{-\tau^2} dt = \sqrt{\pi}\, P(Y > 1.5) = \sqrt{\pi}\,(1 - P(Y \le 1.5)) =$
$= \sqrt{\pi}\,(1 - \Phi(\frac{1.5-0}{\sqrt{0.5}})) = \sqrt{\pi} \cdot 0.0166 = 0.0294$ and hence

$\sum_{\tau\in\mathbb{Z}} e^{-\tau^2} \approx 1 + 2e^{-1} + 2 \cdot 0.0294 = 1.79456$.

For $V(m_n^*) = \frac{1}{n}\lim_{k\to\infty} kV(m_k^*) = \frac{1}{n}\sum_{\tau\in\mathbb{Z}} r_X(\tau) = \frac{1}{n} \cdot 1.79456 < 0.1$, the number of observations $n > \frac{1.79456}{0.1} = 17.9456$, i.e. we need at least 18 observations. □