Solutions to Exam for Random Processes, 5 points/7.5 ECTS

January 13, 2006, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that if g is the inverse Fourier transform of G and $H = g$, then $h(\tau)$ (the inverse Fourier transform of H) is $G(-\tau)$. (3p)

   Solution: We have that $G(-f) = \mathcal{F}(g)(-f) = \int_{\mathbb{R}} e^{i2\pi f\tau} g(\tau)\, d\tau$,
   so $G(-\tau) = \mathcal{F}^{-1}(g)(\tau)$. Now, if $H(f) \equiv g(f)$,
   then $h(\tau) = \mathcal{F}^{-1}(H)(\tau) = \mathcal{F}^{-1}(g)(\tau) = G(-\tau)$. □

2. An elevator has visited the ground floor N_t times by time t (in hours from 8 o'clock in the morning), where $\{N_t : t \in \mathbb{R}^+\}$ is a Poisson process with intensity 3. What is the probability that $N_t + 5 < N_{t+2}$? (3p)

   Solution: $\{N_t\}$ Poi-pr.(3) $\Rightarrow$ $N_{t+2} - N_t \in Poi(3((t+2)-t)) = Poi(6)$.
   $P(N_t+5 < N_{t+2}) = P(N_{t+2}-N_t > 5) = 1-P(N_{t+2}-N_t \le 5) = 1-0.446 = 0.554$. □

3. The weakly stationary Gaussian process $\{X_t : t \in \mathbb{Z}\}$ has cvf

$$r_X(\tau) = \begin{cases} 2 & \text{if } \tau = 0 \\ -1 & \text{if } |\tau| = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate $P(X_t - X_{t+1} < 1)$ (3p)

   (b) In what frequency is the spectral density of $\{X_t\}$ maximal? (3p)

   Solution:

   (a) $E(X_t - X_{t+1}) = E(X_t) - E(X_{t+1}) = m_X - m_X = 0$.
   $V(X_t - X_{t+1}) = V(X_t) + V(X_{t+1}) - 2C(X_t, X_{t+1}) = r_X(0) + r_X(0) - 2r_X(1) =$
   $= 2 + 2 - (-2) = 6$. Now, $\{X_t\}$ is Gaussian so $X_t - X_{t+1} \in N(0,6)$ $\Rightarrow$
   $\Rightarrow$ $P(X_t - X_{t+1} < 1) = \Phi(\frac{1-0}{\sqrt{6}}) = 0.6591$.

   (b) $R_X(f) = \sum_{\tau\in\mathbb{Z}} r_X(\tau) e^{-i2\pi f\tau} = -1 \cdot e^{-i2\pi f\cdot(-1)} + 2 \cdot e^0 - 1 \cdot e^{-i2\pi f\cdot 1} = 2 - 2\cos(2\pi f)$, for $-\frac{1}{2} < f \le \frac{1}{2}$ which is maximal for $f = \frac{1}{2}$ where $R_X(f) = 2 - 2\cdot(-1) = 4$. □

4. Suppose the weakly stationary process $\{X_t\}$ has spectral density $R_X(f) = e^{-\sqrt{|f|}}$, and that $\{X_t\}$ is filtered with impulse response $h(t) = 1$ if $0 \le t \le 1$ and $h(t) = 0$ otherwise. What is the spectral density of the filtered process? (4p)

Solution: $R_X(f) = e^{-\sqrt{|f|}}$ and $h(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$ so the frequency function is $H(f) = \mathcal{F}(h) = \int_0^1 1 \cdot e^{-i2\pi f t} dt = -\frac{1}{i2\pi f}(e^{-i2\pi f} - e^0) = \frac{i}{2\pi f}(e^{-i2\pi f} - 1) = \frac{1}{2\pi f}(\sin(2\pi f) + i(\cos(2\pi f) - 1)) \Rightarrow |H(f)|^2 = \frac{1}{(2\pi f)^2}(\sin^2(2\pi f) + \cos^2(2\pi f) - 2\cos(2\pi f) + 1) = \frac{1-\cos(2\pi f)}{2\pi^2 f^2} \Rightarrow R_Y = |H|^2 R_X = \frac{1-\cos(2\pi f)}{2\pi^2 f^2} e^{-\sqrt{|f|}}$. $\square$

5. Let $\{X_t\}$ be a positively correlated $AR(1)$ process with $\sigma_\epsilon^2 = \frac{1}{4}$ and $\sigma_X^2 = 1$.

(a) Determine the value of the AR parameter a_1. (3p)

(b) Calculate the cvf $r_X(\tau)$ of the process $\{X_t\}$. (3p)

Solution:

(a) $X_t = aX_{t-1} + \epsilon_t$. Taking the variance of both sides yields $1 = \sigma_X^2 = V(X_t) = V(X_{t-1} + \epsilon_t) = a^2 V(X_{t-1}) + V(\epsilon_t) = a^2 \cdot 1 + \frac{1}{4} \Rightarrow a^2 = 1 - \frac{1}{4} = \frac{3}{4}$. Since $\{X_t\}$ was positively correlated we have that $a = \frac{\sqrt{3}}{2}$.

(b) $r_X(1) = C(X_t, X_{t-1}) = C(aX_{t-1} + \epsilon_t, X_{t-1}) = aV(X_{t-1}) = a$.
$r_X(2) = C(X_t, X_{t-2}) = C(a(aX_{t-2} + \epsilon_{t-1}) + \epsilon_t, X_{t-1}) =$
$= C(a^2 X_{t-2} + a\epsilon_{t-1} + \epsilon_t, X_t) = a^2 V(X_{t-2}) = a^2$.
In general $X_t = a^s X_{t-s} + a^{s-1}\epsilon_{t-s+1} + \ldots + a\epsilon_{t-1} + \epsilon_t$ so $C(X_t, X_{t-s}) = C(a^s X_{t-s} + a^{s-1}\epsilon_{t-s+1} + \ldots + a\epsilon_{t-1} + \epsilon_t, X_{t-s}) = a^s V(X_{t-s}) = a^s$ and consequently $r_X(\tau) = a^{|\tau|} = (\frac{\sqrt{3}}{2})^{|\tau|}$. $\square$

6. Suppose the process $\{X_t : t \in \mathbb{R}\}$ which has spectral density function $R_X(f) = e^{-|f|}$ is sampled at time-points $t = 0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \ldots$ What is the spectral density of the sampled signal. (4p)

Solution:[1] $R_Y(f) = \sum_{k\in\mathbb{Z}} R_X(f + \frac{k}{1/3})$ {where $-\frac{1}{2/3} < f \le \frac{1}{2/3}$} $=$

$$= \sum_{k\in\mathbb{Z}} R_X(f + 3k) \quad \{\text{where } -\tfrac{3}{2} < f \le \tfrac{2}{3}\} = \sum_{k=-\infty}^{\infty} e^{-|f+3k|} =$$

$$= \sum_{k=-\infty}^{-1} e^{-|f+3k|} + e^{-|f|} + \sum_{k=1}^{\infty} e^{-|f+3k|} = \sum_{k=1}^{\infty} e^{-|f-3k|} + e^{-|f|} + \sum_{k=1}^{\infty} e^{-|f+3k|} =$$

$$= \sum_{k=1}^{\infty} e^{f-3k} + e^{-|f|} + \sum_{k=1}^{\infty} e^{-(f+3k)} = e^f \sum_{k=1}^{\infty} e^{-3k} + e^{-|f|} + e^{-f} \sum_{k=1}^{\infty} e^{-3k} =$$

$$= e^{-|f|} + (e^f + e^{-f}) \sum_{k=1}^{\infty} (e^{-3})^k = e^{-|f|} + (e^f + e^{-f}) \cdot \frac{e^{-3}}{1-e^{-3}} = e^{-|f|} + \frac{(e^{|f|} + e^{-|f|})e^{-3}}{1-e^{-3}} \quad \square$$

[1] Thanks to Li Rui for correcting comment!

7. Let $Y_t = \sqrt{e^{W_t^2}}$ where $\{W_t\}$ is a standard Wiener process. Calculate the density function of Y_t. (4p)

Solution: Begin by observing that $W_t^2 \geq 0 \;\Rightarrow\; e^{W_t^2} \geq 1 \;\Rightarrow\; \sqrt{e^{W_t^2}} \geq 1$. Then the density function of Y_t is

$$
\begin{aligned}
f_{Y_t}(y) &= \tfrac{d}{dy} P(Y_t \leq y) \\
&= \tfrac{d}{dy} P(\sqrt{e^{W_t^2}} \leq y) \\
&= \tfrac{d}{dy} P(W_t^2 \leq \ln y^2) \\
&= \tfrac{d}{dy} P(-\sqrt{2\ln y} \leq W_t \leq \sqrt{2\ln y}) \quad \text{(ok since } Y_t = \sqrt{e^{W_t^2}} \geq 1.\text{)} \\
&= \tfrac{d}{dy}\left(\Phi\left(\sqrt{\tfrac{2\ln y}{t}}\right) - \Phi\left(-\sqrt{\tfrac{2\ln y}{t}}\right)\right) \\
&= \tfrac{d}{dy}\left(2\Phi\left(\sqrt{\tfrac{2\ln y}{t}}\right) - 1\right) \\
&= 2 \cdot \frac{\frac{1}{t} \cdot \frac{1}{y^2} \cdot 2y}{2\sqrt{\frac{2}{t}\ln y}} \cdot \phi\left(\sqrt{\tfrac{2\ln y}{t}}\right) \\
&= \frac{2}{y\sqrt{2t\ln y}}\,\phi\left(\sqrt{\tfrac{2\ln y}{t}}\right) \\
&= \frac{2}{y\sqrt{2t\ln y}} \cdot \frac{1}{\sqrt{2\pi}}\, e^{-(\sqrt{2\ln y/t})^2/2} \\
&= \frac{1}{y\sqrt{\pi t\ln y}}\, e^{-\frac{1}{t}\ln y} \\
&= \frac{1}{y\sqrt{\pi t\ln y}}\, e^{\ln y^{-1/t}} \\
&= \frac{1}{y^{1+1/t}\sqrt{\pi t\ln y}}
\end{aligned}
$$

□