Solutions to Exam for Random Processes, 5 p/7.5 ECTS

August 18, 2006, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume $\{X_t\}$ is a weakly stationary process which is filtered with a linear filter with transfer function H resulting in the output $\{Y_t\}$. Show that $V(Y_t) = \int_{\mathbb{R}} |H(f)|^2 R_X(f)\,df$. (4p)

   Solution: See page 84 in *Random processes* by Lindgren and Rootzén. □

2. Suppose that $\{X_t\}$ is a Poisson process with intensity 4 and let $Y_t = \frac{1}{2}(X_t - 4t)$. Calculate

   (a) $C(X_t, X_{t+1})$. (3p)

   (b) $P(Y_t > Y_{t+1})$. (4p)

   Solution:

   (a) $C(X_t, X_{t+1}) = C(X_t, X_{t+1} - X_t + X_t) = C(X_t, X_{t+1} - X_t) + C(X_t, X_t) = 0 + 4 = 4$

   (b) $P(Y_t > Y_{t+1}) = P(Y_{t+1} - Y_t < 0) = P(\frac{1}{2}(X_{t+1} - 4(t+1)) - \frac{1}{2}(X_t - 4t) < 0) = P(X_{t+1} - X_t < 2(4(t+1) - 4t)) = P(X_{t+1} - X_t = 0) + P(X_{t+1} - X_t = 1) + P(X_{t+1} - X_t = 2) + P(X_{t+1} - X_t = 3) = \frac{4^0}{0!}e^{-4} + \frac{4^1}{1!}e^{-4} + \frac{4^2}{2!}e^{-4} + \frac{4^3}{3!}e^{-4} = 0.433$ (which also could have been read from the tables!) □

3. Can

   $$R(f) = \begin{cases} 1 + \cos(\pi f) & \text{if } -1 \le f < 1 \\ 0 & \text{otherwise} \end{cases}$$

   be the spectral density function of a weakly stationary process in continuous time? Motivate your answer. (3p)

   Solution: No, because the spectral density function has finite support (i.e. $R(f) = 0$ outside a finite interval, in this case outside $\{f \in \mathbb{R} : -1 \le f < 1\}$) which means that the random process must be in discrete time (at time-points $t = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots\}$ as a matter of fact), not a process in continuous time. □

4. Assume that $\{\epsilon_t\}$ is normally distributed white noise with $\sigma_\epsilon^2 = 1$ and that the process $\{X_t : t \in \mathbb{Z}\}$ is defined by $X_t = \epsilon_t \epsilon_{t-1}$ for all t.

   (a) Is the white noise process, $\{\epsilon_t\}$, strongly stationary? (2p)

   (b) Show that $\{X_t\}$ is a sequence of uncorrelated variables. (4p)

Solution:

(a) Since $\{\epsilon_t\}$ is white noise with $\sigma_\epsilon^2 = 1$ we have that $E(\epsilon_t) = 0$, $V(\epsilon_t) = 1$ and $\epsilon_s \perp \epsilon_t$ for all $s \neq t$. In addition we have normal distribution so $\epsilon_t \in N(0,1)$ for all $t \in \mathbb{Z}$. Thus the joint density function of $\{X_{t_1}, \ldots, X_{t_n}\}$ is $\prod_{k=1}^n \phi(x_{t_k})$ and the joint density function of $\{X_{t_1+h}, \ldots, X_{t_n+h}\}$ is $\prod_{k=1}^n \phi(x_{t_k+h})$, i.e. the density (and therefore the distribution) does not change as time shifts uniformly for all varibles, which is to say that the process is strongly stationary.

(b) $E(X_t) = E(\epsilon_t)E(\epsilon_{t-1}) = 0 \Rightarrow C(X_t, X_{t+h}) = E(\epsilon_t \epsilon_{t-1} \epsilon_{t+h} \epsilon_{t+h-1})$ so

$$\begin{aligned}
h = 0: &\quad C(X_t, X_{t+h}) = E(\epsilon_t^2)E(\epsilon_{t-1}^2) = V(\epsilon_t)V(\epsilon_{t-1}) = 1.\\
h = 1: &\quad C(X_t, X_{t+h}) = E(\epsilon_t^2)E(\epsilon_{t-1})E(\epsilon_{t+1}) = 0.\\
h = -1: &\quad C(X_t, X_{t+h}) = E(\epsilon_{t-1}^2)E(\epsilon_t)E(\epsilon_{t-2}) = 0.\\
|h| > 1: &\quad C(X_t, X_{t+h}) = E(\epsilon_t)E(\epsilon_{t-1}E(\epsilon_{t+h})E(\epsilon_{t+h-1}) = 0.
\end{aligned}$$

Thus the process $\{X_t\}$ is uncorrelated. □

5. An $AR(1)$ process, $\{X_t\}$, is realised at consecutive timepoints:

   $x_1 = -0.54 \quad x_2 = 0.22 \quad x_3 = 0.07 \quad x_4 = -2.20 \quad x_5 = 0.75 \quad x_6 = -1.01 \quad x_7 = 0.58$

   By using these observations, estimate the covariance between X_t and X_{t-1}. (4p)

Solution: Since $E(X_t) = 0$ in an $AR(1)$ process, using the estimator $\hat{r}(\tau) = \frac{1}{n}\sum_{k=1}^{n-\tau} X_k X_{k+\tau}$, we get $\hat{r}(1) = \frac{1}{7}\sum_{k=1}^{6} x_k x_{k+1} = \frac{1}{7}(-0.1188 + 0.0154 - 0.154 - 1.65 - 0.7575 - 0.5858) = -0.4644$. □

6. Let $\{X_t : t \in \mathbb{R}\}$ be a weakly stationary process with $m_X = 1$ and $r_X(\tau) = \max(0, 1 - |\tau|)$. Further let the process $\{Y_t\}$ be defined by $Y_t = \int_0^t X_s\, ds$. Determine the spectral density, $R_Y(f)$, of the output signal. (6p)

Solution: $Y_t = \int_{t-1}^t X_s\, ds = \int_{\mathbb{R}} h(s) X_{t-s}\, ds = \left\{ \begin{array}{l} u = t - s \\ du = -ds \end{array} \right\} = \int_{\infty}^{-\infty} h(t-u) X_u\, (-du) = \int_{\mathbb{R}} h(t-u) X_u\, du = \int_{t-1}^t X_u\, du$ if $h(t-u) = \begin{cases} 1 & \text{if } t-1 \leq t-u \leq t \\ 0 & \text{otherwise} \end{cases} \Rightarrow h(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$
$\Rightarrow H(f) = \int_{\mathbb{R}} h(s) e^{-i2\pi f s} ds = \int_0^1 e^{-i2\pi f s} ds = [-\frac{1}{i2\pi f} e^{-i2\pi f s}]_0^1 = \frac{1 - e^{-i2\pi f}}{i2\pi f} = -i\frac{1}{2\pi f}(1 - \cos(2\pi f)) + \frac{1}{2\pi f}\sin(2\pi f) \Rightarrow |H|^2 = \frac{1}{(2\pi f)^2}((1 - \cos(2\pi f))^2 + \sin^2(2\pi f)) = \frac{1}{4\pi^2 f^2}(1 - 2\cos(2\pi f) + \cos^2(2\pi f) + \sin^2(2\pi f)) = \begin{cases} \frac{1 - \cos(2\pi f)}{2\pi^2 f^2} & \text{if } f \neq 0 \\ 1 & \text{otherwise} \end{cases}$. Finally $r_X = \max(0, 1 - |\tau|)$ implies (according to the tables) that also $R_X = \begin{cases} \frac{1 - \cos(2\pi f)}{2\pi^2 f^2} & \text{if } f \neq 0 \\ 1 & \text{otherwise} \end{cases}$ and hence $R_Y = |H|^2 R_X = \begin{cases} (\frac{1 - \cos(2\pi f)}{2\pi^2 f^2})^2 & \text{if } f \neq 0 \\ 1 & \text{otherwise} \end{cases}$ □