Solutions to Exam for Random Processes, 5 points/7.5 ECTS

April 21, 2006, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53).

1. Show that if $\{X_t\}$ is weakly stationary, H is a transfer function for a linear filter, $\{Y_t\}$ is the filtered process and R_X and R_Y are the respective spectral densities. then $R_Y(f) = |H(f)|^2 R_X(f)$. (4p)

   Solution: (See p. 84 in the book *Random processes* by Rootzén & Lindgren.) □

2. For a stationary process in continuous time, $\{X_t\}$, and its derivative in squared mean, $\{X_t'\}$, show that $C(X_t, X_t') = 0$. (3p)

   Solution: (See p. 101 in the book *Random processes* by Rootzén & Lindgren.) □

3. Can $r(\tau) = \frac{1}{1+4\pi^2(1-\tau)^2} + \frac{1}{1+4\pi^2(1+\tau)^2}$ be cvf of a weakly stationary process? (3p)

   Solution: If the cvf of $\{X_t\}$ is as given above and $\{X_t\}$ is weakly stationary, then the spectral density function is $R_X(f) = e^{-|f|}\cos(2\pi f)$ (according to the table read "backwards" with $\alpha = 1$ and $\tau_0 = 1$). But then $R_X(f) < 0$ for some f, e.g. for $|f| \in (\frac{1}{4}, \frac{3}{4})$ which is not allowed. Thus r_X according to above cannot be a cvf of a weakly stationary process. □

4. Let the process $\{X_t\}$ be shot noise with intensity $\ln 2$ and impulse function

$$g(t) = \begin{cases} 2^{-t} & \text{for } t \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

   Determine approximately the probability $P(X_t > 1 + X_{t-1})$. (4p)

Solution: $P(X_t > 1 + X_{t-1}) = 1 - P(X_t - X_{t-1} \le 1) = 1 - \Phi(\frac{1-\mu}{\sigma})$, where $\mu = E(X_t - X_{t-1})$ and $\sigma = \sqrt{V(X_t - X_{t-1})}$. According to the Campbell formulae $E(X_t) = \ln 2 \cdot \int_0^\infty 2^{-t}\,dt = \ln 2 \cdot \int_0^\infty e^{-t\ln 2}\,dt = \ln 2 \cdot [-\frac{e^{-t\ln 2}}{\ln 2}]_0^\infty = 1$ and $r(\tau) = \ln 2 \cdot \int_{\mathbb{R}} g(u)\,g(u-\tau)\,du \overset{\tau\ge 0}{=} \ln 2 \cdot \int_\tau^\infty 2^{-u}2^{-(u-\tau)}\,du = \ln 2 \cdot 2^\tau \int_\tau^\infty e^{-2u\ln 2}\,du = \ln 2 \cdot 2^\tau[-\frac{e^{-2u\ln 2}}{2\ln 2}]_\tau^\infty = \ln 2 \cdot 2^\tau(0 + \frac{e^{-2\tau\ln 2}}{2\ln 2}) = 2^{-\tau-1}$. Therefore we have that $\mu = E(X_t - X_{t-1}) = E(X_t) - E(X_{t-1}) = 0$ and $\sigma = \sqrt{V(X_t - X_{t-1})} = \sqrt{V(X_t) + V(X_{t-1}) - 2C(X_t, X_{t-1})} = \sqrt{2^{-1} + 2^{-1} - 2^{-2}} = 0.86603$. Thus we finally get $P(X_t > 1 + X_{t-1}) = 1 - \Phi(\frac{1-0}{0.86603}) = 1 - \Phi(1.15) = 0.1251$. □

5. The observations 1.2, 5.3, 1.7, 0.8, 3.1 of a strongly stationary process $\{X_t\}$ are made. Estimate the cvf of $\{X_t\}$ from this sample. (3p)

Solution: Choosing the estimator $r_n^* = \frac{1}{n}\sum_{t=1}^{n-\tau}(x_t - \bar{x})(x_{t+\tau} - \bar{x})$ we get $\bar{x} = 2.42$ so $x_1 - \bar{x} = -1.22$, $x_2 - \bar{x} = 2.88$, $x_3 - \bar{x} = -0.72$, $x_4 - \bar{x} = -1.62$, $x_5 - \bar{x} = 0.68$, so

$$r_5^*(\tau) = \begin{cases} \frac{1}{5}((x_1-\bar{x})^2 + \ldots + (x_5-\bar{x})) & \text{if } \tau = 0 \\ \frac{1}{5}((x_1-\bar{x})(x_2-\bar{x}) + \ldots + (x_4-\bar{x})(x_5-\bar{x})) & \text{if } |\tau| = 1 \\ \frac{1}{5}((x_1-\bar{x})(x_3-\bar{x}) + \ldots + (x_3-\bar{x})(x_5-\bar{x})) & \text{if } |\tau| = 2 \\ \frac{1}{5}((x_1-\bar{x})(x_4-\bar{x}) + (x_2-\bar{x})(x_5-\bar{x})) & \text{if } |\tau| = 3 \\ \frac{1}{5}(x_1-\bar{x})(x_5-\bar{x}) & \text{if } |\tau| = 4 \\ 0 & \text{if } |\tau| \ge 5 \end{cases} = \begin{cases} 2.6776 & \text{if } \tau = 0 \\ -1.10448 & \text{if } |\tau| = 1 \\ -0.85536 & \text{if } |\tau| = 2 \\ 0.78696 & \text{if } |\tau| = 3 \\ -0.16592 & \text{if } |\tau| = 4 \\ 0 & \text{if } |\tau| \ge 5 \end{cases}$$
□

6. Let $\{X_t\}$ be an $AR(1)$ process with parameter $\frac{1}{2}$, i.e. $X_t = -0.5X_{t-1} + \epsilon_t$ for all $t \in \mathbb{Z}$ where $\{\epsilon_t\}$ is normally distributed white noise with variance $\sigma_\epsilon^2 = 3$.

   (a) Calculate $C(X_1, X_2)$. (4p)

   Let $\{Y_t\}$ be an $AR(1)$ process with parameter $-\frac{1}{5}$, i.e. $Y_t = 0.2Y_t + \eta_t$ where $\{\eta_t\}$ is normally distributed white noise with variance $\sigma_\eta^2 = 1$ independently of $\{\epsilon_t\}$.

   (b) Determine numbers $\alpha, \beta \in \mathbb{R}$ such that $\{Z_t\}$ is an $AR(1)$ process where $Z_t = \alpha X_t + \beta Y_{t-1}$ for all $t \in \mathbb{Z}$ and calculate the variance $\{Z_t\}$. (5p)

Solution:

(a) $\sigma_X^2 = V(X_t) = V(-\frac{1}{2}X_{t-1} + \epsilon_t) = \frac{1}{4}V(X_{t-1}) + V(\epsilon_t) = \frac{1}{4}\sigma_X^2 + 3 \Rightarrow \frac{3}{4}\sigma_X^2 = 3 \Rightarrow \sigma_X^2 = 4 \Rightarrow C(X_1, X_2) = C(X_1, -\frac{1}{2}X_1 + \epsilon_2) = -\frac{1}{2}V(X_1) + C(X_1, \epsilon_2) = -\frac{1}{2}\sigma_X^2 = -2$.

(b) Obviously we can choose $\alpha = 0$ and $\beta \neq 0$ or $\alpha \neq 0$ and $\beta = 0$, which in either case will make $\{Z_t\}$ an $AR(1)$ process. □

7. Let $\{N(t) : t \geq 0\}$ be a Poisson process with intensity 1 and let the process $\{Y_t\}$ be defined by $Y_t = e^{-t}N(e^{2t}) - e^t$ for all $t \in \mathbb{R}^+$. Show that $\{Y_t\}$ is weakly stationary. (4p)

Solution:

$$\begin{aligned} E(Y_t) &= e^{-t}E(N(e^{-2t})) - e^t \\ &= e^{-t} \cdot 1 \cdot e^{2t} - e^t \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} C(Y_s, Y_t) &= C(e^{-s}N(e^{2s}) - e^s, e^{-t}N(e^{2t}) - e^t) \\ &= e^{-s-t}C(N(e^{2s}), N(e^{2t})) \\ &= e^{-s-t}1 \cdot \min(e^{2s}, e^{2t}) \\ &= e^{-s-t}e^{\min(2s,2t)} \\ &= e^{\min(2s-s-t,2t-s-t)} \\ &= e^{\min(s-t,t-s)} \\ &= e^{-\max(t-s,s-t)} \\ &= e^{-|s-t|} \end{aligned}$$

Since $E(Y_t)$ is constant with respect to t and $C(Y_s, Y_t)$ is only a function of s and t by the difference $s - t$, we have proved that the process $\{Y_t\}$ is weakly stationary. □