4) Diskret spektrum (diskret tid)

$$r(\tau) = \sum_k A_k \cos(2\pi f_k t + \phi_k) \quad \text{där } 0 \le f_k \le \frac{1}{2}$$

enl. Eulers formel

samband mellan vvf, kvf, spektr.

Sats 6 Vikningseffekten (s. 75)

Om $\{Y_t : t \in \mathbb{R}\}$ svagt stationär

observeras i tidpunkterna

$t = \ldots, -2d, -d, 0, d, 2d, \ldots$

$Z_t = Y_t$ då $t = 0, \pm d, \pm 2d, \ldots$

så $m_Z = m_Y$ och $r_Z = r_Y$

där $r_Z(\tau) = \displaystyle\int_{-\frac{1}{2d}}^{\frac{1}{2d}} e^{i2\pi f \tau} R_Z(f)\,df \qquad \tau = 0, \pm d, \pm 2d, \ldots$

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y\left(f + \frac{k}{d}\right) \qquad -\frac{1}{2d} < f \le \frac{1}{2d}$$

Bevis av sats 6:

Antag $\{Y_t\}$ svagt stationär och att

$Z_t := Y_t$ för $t \in \{nd : n \in \mathbb{Z}\}$.

Då är

$$r_Z(\tau) = r_Y(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f \tau} R_Y(f)\, df =$$

$$\overset{\{\tau = nd\}}{=} \int_{-\infty}^{\infty} e^{i2\pi f n d} R_Y(f)\, df$$

$\{$Dela upp $(-\infty, \infty)$ $\ldots, \left(-\frac{3}{2d}, -\frac{1}{2d}\right], \left(-\frac{1}{2d}, \frac{1}{2d}\right], \left(\frac{1}{2d}, \frac{3}{2d}\right], \ldots\}$

$$= \sum_{k=-\infty}^{\infty} \int_{\frac{2k-1}{2d}}^{\frac{2k+1}{2d}} e^{i2\pi f n d} R_Y(f)\, df$$

$$\left\{ \begin{array}{lll} \text{Substitution} & f = (2k-1)/(2d) & f = (2k+1)/(2d) \\ f = v + k/d & v = f - k/d & v = f - k/d \\ df = dv & \quad = \frac{2k-1}{2d} - \frac{k}{d} & \quad = \frac{2k+1-2k}{2d} \\ & \quad = \frac{2k-1-2k}{2d} = -\frac{1}{2d} & \quad = \frac{1}{2d} \end{array} \right\}$$

$$= \sum_{k=-\infty}^{\infty} \int_{-1/2d}^{1/2d} e^{i2\pi(v + k/d)nd} R_Y(v + k/d)\, dv$$

$$= \sum_{k=-\infty}^{\infty} \int_{-1/2d}^{1/2d} e^{i2\pi v n d} \underbrace{e^{i2\pi \underbrace{kn}_{\text{heltal}}}}_{=1} R_Y(v + k/d)\, dv$$

$$= \int_{-1/2d}^{1/2d} \sum_{k=-\infty}^{\infty} e^{i2\pi v n d} R_Y(v + k/d)\, dv$$

$$= \int_{-1/2d}^{1/2d} e^{i2\pi v \underbrace{nd}_{\tau}} R_Z(v)\, dv \quad \text{där } R_Z(v) = \sum_{k=-\infty}^{\infty} R_Y\left(v + \frac{k}{d}\right)$$

$\square$

Ex Antag $\{X_t\}$ har spektraltäthet

$$R_X(f) = \begin{cases} c & |f| \le 1 \\ 0 & \text{annars} \end{cases}$$ och att

$Z_k = X_k$ där $k = 0, \pm d, \pm 2d, \ldots$

Beräkna R_Z och r_Z.

$$R_Z = \sum_{k=-\infty}^{\infty} R_X\left(f + \frac{k}{d}\right)$$

$$\left\{ \begin{array}{ll} \left|f + \frac{k}{d}\right| \le 1 & -1 \le f + \frac{k}{d} \le 1 \\ d(-1-f) \le k \le d(1-f) & \end{array} \right\}$$

$$= \sum_{k=-d(1+f)}^{d(1-f)} c$$

$$= \big(d(1-f) - (-d(1+f))\big)\, c$$

$$= (1 - f + 1 + f)\, cd$$

$$= 2cd \quad \text{då} \quad -\frac{1}{2d} < f \le \frac{1}{2d}$$

Eftersom $r_z = r_x$ har vi

$$r_z = r_x = \int_{-\infty}^{\infty} e^{i2\pi f\tau} R_x(f)\,df =$$

$$= \int_{-1}^{1} e^{i2\pi f\tau} c\,df = c\left(\int_{-1}^{0} e^{i2\pi f\tau} df + \int_{0}^{1} e^{i2\pi f\tau} df\right) =$$

$$= c\left(\int_{1}^{0} e^{i2\pi(-f)\tau}(-df) + \int_{0}^{1} e^{i2\pi f\tau} df\right)$$

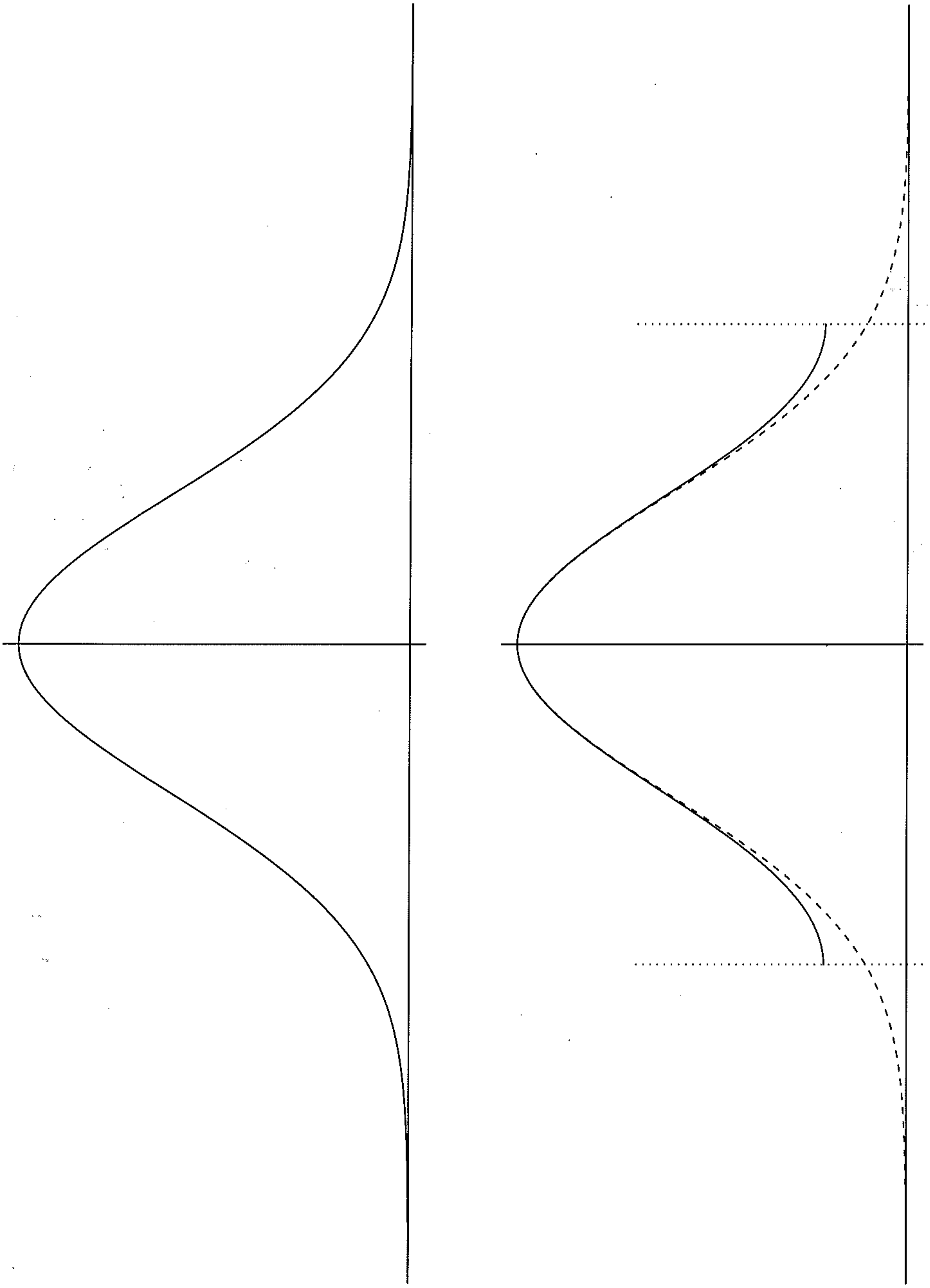
$$= c\left(\int_{0}^{1} e^{-i2\pi f\tau} df + \int_{0}^{1} e^{i2\pi f\tau} df\right)$$

$$= c\cdot 2\int_{0}^{1} \frac{e^{-i2\pi f\tau} + e^{i2\pi f\tau}}{2}\,df$$

$$= 2c\int_{0}^{1} \cos(2\pi f\tau)\,df$$

$$= 2c\left[\frac{\sin(2\pi f\tau)}{2\pi\tau}\right]_{0}^{1} = 2c\left(\frac{\sin(2\pi\tau)}{2\pi\tau} - 0\right)$$

$$= \frac{c\sin(2\pi\tau)}{\pi\tau} \quad \text{där } \tau = 0, \pm d, \pm 2d, \ldots$$



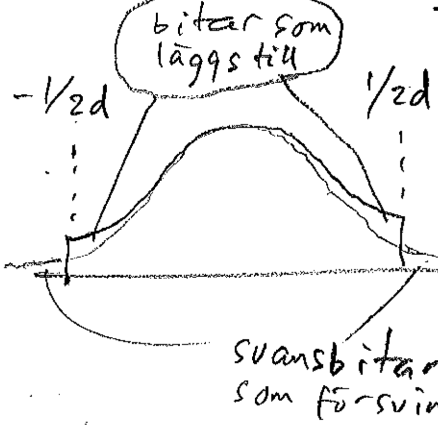
Hur ska man kontrollera vikningseffekten?

Genom att välja tätare samplingspkter får man med mer av processen och reducerar distorsionsrisk pga vikningseff.

Men hur tätt ska samplingspkterna väljas för en given riskmarginal $\alpha \cdot 100\%$.

Antag att processen är svagt stationär och att vvf $m = 0$. Då är

$$\text{signaleffekten} = E(X_t^2) = V(X_t) =$$
$$= r(0) = \int_{-\infty}^{\infty} e^{i2\pi f \cdot 0} R(f)\,df = \int_{-\infty}^{\infty} R(f)$$



Tot: 4 bitar!

4 svansbitar $\leq$ $\alpha \cdot$ hela signaleffekten

Intervallängden fås genom att lösa ut d ur

$$4\int_{1/2d}^{\infty} R(f)\,df \leq \alpha \int_{-\infty}^{\infty} R(f)\,df$$

EX En svagt stationär signal har spektrum

$$R(f) = e^{-2\pi|f|}$$

Hur tätt ska man sampla för att högst 1 procent av signalen ska distorderas?

$$\int R(t)\,df = \int e^{-2\pi|f|}\,df$$

Minns $\underbrace{\text{4 svansbitar}} \leq \underbrace{\alpha \text{ hela signalen}}$

$$= 4\int_{1/2d}^{\infty} e^{-2\pi|f|}\,df \qquad = \alpha \int_{-\infty}^{\infty} e^{-2\pi|f|}\,df$$

$$= 4\left[-\frac{e^{-2\pi f}}{2\pi}\right]_{1/2d}^{\infty} \qquad = 2\alpha\int_0^{\infty} e^{-2\pi f}\,df$$

$$= 4\left(-0 + \frac{e^{-\pi/d}}{2\pi}\right) \qquad = 2\alpha\left[-\frac{e^{-2\pi f}}{2\pi}\right]_0^{\infty}$$

$$= 2\cancel{4}e^{-\pi/d}/\pi \qquad = 2\alpha\left(-0 + \frac{1}{2\pi}\right)$$

$$= \alpha/\pi$$

$$\frac{2\cancel{4}e^{-\pi/d}}{\cancel{\pi}} \leq \frac{\alpha}{\cancel{\pi}}$$

$$e^{-\pi/d} \leq \frac{\alpha}{\cancel{4}2}$$

$$-\frac{\pi}{d} \leq \log\frac{\alpha}{\cancel{4}2}$$

$$\frac{\pi}{d} \geq -\log\frac{\alpha}{\cancel{4}2} = \log\frac{\cancel{4}2}{\alpha}$$

$$d \leq \frac{\pi}{\log\frac{\cancel{4}2}{\alpha}} = \frac{\pi}{\log\frac{\cancel{4}2}{0.01}} = \cancel{0.924}\ 0.59$$

Ex (tentauppg?)

Hur tätt sampla för α distorderas?

$$\underbrace{4\int_{1/2d}^{\infty} R(f)\,df}\ \le\ \underbrace{\alpha\int_{-\infty}^{\infty} R(f)\,df}$$

$$= 4\int_{1/2d}^{\infty} \frac{df}{1+f^2} \qquad\qquad = \alpha\,\big[\arctan f\big]_{-\infty}^{\infty}$$

$$= 4\,\big[\arctan f\big]_{1/2d}^{\infty} \qquad\qquad = \alpha\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \pi\alpha$$

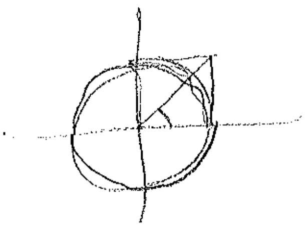
$$= 4\left(\frac{\pi}{2} - \arctan\frac{1}{2d}\right)$$

$$= 2\pi - 4\arctan\frac{1}{2d}$$

$$\frac{200\pi}{400} \geqslant \frac{\pi}{2}$$

$$2\pi - 4\arctan\frac{1}{2d} \le \alpha\pi$$

$$\pi(2-\alpha) \le 4\arctan\frac{1}{2d}$$



$$\tan\frac{\pi}{4}(2-\alpha) \le \frac{1}{2d}$$

$$d \le \left(\tan\frac{\pi}{4}(2-\alpha)\right)^{-1}$$

t.ex. med $\alpha = 10\%$, $d \le 0.0787$