Solutions to Exam for Random Processes, 7.5 ECTS

August 12, 2009, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show that the spectral density of an AR(p) process is

$$R(f) \;=\; \frac{\sigma_\epsilon^2}{|1+\sum_{k=1}^{p} a_k e^{-i2\pi fk}|^2} \qquad \text{(4p)}$$

Solution: (See p 98 in the course literature *Random processes*.) □

2. Let $\{N_t : t \in \mathbb{R}^+\}$ be a Poisson process with intensity 3, and let $\{Y_t : t \in \mathbb{R}^+\}$ be defined by $Y_t = N_t + N_{t+2}$. What is the covariance function of Y_t? (3p)

Solution: $C(Y_s, Y_t) = C(N_s + N_{s+2}, N_t + N_{t+2}) =$
$= 3\Big(\min(s,t) + \min(s,t+2) + \min(s+2,t) + \min(s+2,t+2)\Big) =$

$$= \begin{cases} 3(s+s+s+2+s+2) & s+2 \le t \\ 3(s+s+t+s+2) & t < s+2 \le t+2 \\ 3(t+s+t+t+2) & s \le t+2 < s+2 \\ 3(t+t+2+t+t+2) & t+2 < s \end{cases} = \begin{cases} 12s+12 & s+2 \le t \\ 9s+2t+6 & t < s+2 \le t+2 \\ 3s+9t+6 & s \le t+2 < s+2 \\ 12t+12 & t+2 < s \end{cases}$$

□

3. Calculate the covariance function of a shot noise process with parameter 0.05 and pulse function $g(t) = I(t \in [0,1])$. (4p)

Solution: Assume $\tau > 0$ (and $\tau < 1$). Then $r(\tau) = \lambda \int g(u)g(u-\tau)\,du = 0.05 \int_{\mathbb{R}} \underbrace{I(u \in [0,1])}_{=1 \text{ if } 0 \le u \le 1} \underbrace{I(u-\tau \in [0,1])}_{=1 \text{ if } 0 \le u-\tau \le 1} du = 0.05 \int_\tau^1 1 \cdot 1\,du = 0.05(1-\tau)$. When $\tau > 1$ the intervals $0 \le u \le 1$ and $\tau \le u \le 1+\tau$ are disjoint implying that $I(u \in [0,1])I(u-\tau \in [0,1]) = 0$ and consequently $r(\tau) = 0$ for $\tau > 1$. Similar reasoning for $-1 < \tau \le 0$ and $\tau \le -1$ respectively. Thus we have totally that $r(\tau) = 0.05\max(0, 1-|\tau|)$. □

4. Assume that the weakly stationary process $\{X_t\}$ is observed to be

$$10.3 \quad 11.2 \quad 10.9 \quad 9.1 \quad 9.3$$

at times $t = 1, 2, 3, 4, 5$, and that it is known that $E(X_t) = 10$. Determine an unbiased estimator of the covariance function of $\{X_t\}$. (3p)

Solution: The unbiased estimator (found in the summary of formulae)

$$\hat{r}(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m)(X_{t+\tau} - m)$$

becomes in this case:

$\hat{r}(0) = \frac{1}{5}(0.09 + 1.69 + 0.81 + 0.81 + 0.49) = 0.778$

$\hat{r}(\pm 1) = \frac{1}{5}(0.39 + 1.17 - 0.81 + 0.63) = 0.276$

$\hat{r}(\pm 2) = \frac{1}{5}(0.27 - 1.17 - 0.63) = -0.306$

$\hat{r}(\pm 3) = \frac{1}{5}(-0.27 - 0.91) = -0.236$

$\hat{r}(\pm 4) = \frac{1}{5}(-0.21) = -0.042$

and $\hat{r}(\tau) = 0$ for $|\tau| \geq 5$. □

5. Assume that $\{X_t : t \in \mathbb{R}\}$ is weakly stationary with covariance function $g(a\tau - b)$ where $a \neq 0$. Determine the spectral density of $\{X_t\}$ in terms of $G = \mathcal{F}(g)$. (4p)

Solution: Since $G(x) = \int e^{-i2\pi xy} g(y)\, du$, we have that

$$\begin{aligned} R(f) &= \int_{\mathbb{R}} e^{-i2\pi f\tau} g(a\tau - b)\, d\tau \quad \left\{ \begin{array}{rcl} u &=& a\tau \\ du &=& a\, d\tau \end{array} \right\} \\ &= \int_{\mathbb{R}} e^{-i2\pi f\frac{u}{a}} g(u-b)\frac{1}{a}\, du \\ &= \frac{1}{a}\int_{\mathbb{R}} e^{-i2\pi \frac{f}{a}u} g(u-b)\, du \quad \left\{ \begin{array}{rcl} v &=& u-b \\ dv &=& du \end{array} \right\} \\ &= \frac{1}{a}\int_{\mathbb{R}} e^{-i2\pi \frac{f}{a}(v+b)} g(v)\, dv \\ &= \tfrac{1}{a}e^{-i2\pi\frac{f}{a}b}\int_{\mathbb{R}} e^{-i2\pi\frac{f}{a}v} g(v)\, dv \\ &= \tfrac{1}{a}e^{-i2\pi fb/a}G(\tfrac{f}{a}) \end{aligned}$$

□

6. Consider the weakly stationary process $\{\theta_t : t \in \mathbb{R}\}$ with spectral density $R(f) = e^{-f^2}$. How closely should observations of $\{\theta_t\}$ be sampled to make the risk for aliasing smaller than 5%? (4p)

Solution: By sampling the spectral density of the continuous signal is transformed into the spectral density of the sampled discrete signal. Thus the spectral density is truncated outside the interval $[-\frac{1}{2d}, \frac{1}{2d}]$, and also the density in these tails is added to the part inside the interval $[-\frac{1}{2d}, \frac{1}{2d}]$ for the variance to remain the same. Thus the error is approximately $4\int_{1/(2d)}^{\infty} R(f)\,df = 4\int_{1/(2d)}^{\infty} e^{-f^2} df = 4\int_{1/(2d)}^{\infty} e^{-\frac{1}{2}(\sqrt{2}f)^2} df \left\{ \begin{array}{ll} v = \sqrt{2}f & f = 1/(2d) \\ dv = \sqrt{2}df & v = \frac{1}{\sqrt{2}d} \end{array} \right\} = 4\int_{1/(\sqrt{2}d)}^{\infty} \sqrt{2\pi}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} \frac{1}{\sqrt{2}}\,dv = 4\sqrt{\pi} \int \int_{1/(\sqrt{2}d)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv = 4\sqrt{\pi}(1 - \Phi(\frac{1}{\sqrt{2}d})) < 0.05 \Rightarrow 1 - \Phi(\frac{1}{\sqrt{2}d})) < \frac{0.05}{4\sqrt{\pi}} \Rightarrow \Phi(\frac{1}{\sqrt{2}d}) > 1 - \frac{0.05}{4\sqrt{\pi}} \Rightarrow \frac{1}{\sqrt{2}d} > \Phi^{-1}(1 - \frac{0.05}{4\sqrt{\pi}})$. From the tables we get that $\Phi^{-1}(0.9960) = 2.65 \Rightarrow d < \frac{\sqrt{2}\cdot 2.65}{=}0.2668$. Thus the samples should be made at distance less than 0.2668 apart. □

7. A strongly stationary Gaussian process $\{X_t : t \in \mathbb{R}\}$ has covariance function $r(\tau) = e^{-|\tau|}$.

   (a) Calulate $P(X_t - X_{t+0.5} > 1)$. (3p)

   (b) Assume that $\{X_t\}$ is filtered with transfer function

$$H(f) = \left\{ \begin{array}{ll} (1 + (2\pi f)^2)^{-1/2} & |f| < 1 \\ 0 & \text{otherwise} \end{array} \right.$$

   Show that 4 is an upper bound for the variance of the filtered signal. (5p)

Solution:

(a) $\{X_t\}$ Gaussian $\Rightarrow$ $X_t - X_{t+0.5}$ is normally distributed. $\{X_t\}$ stationary $\Rightarrow E(X_t) = m \Rightarrow E(X_t - X_{t+0.5}) = 0$ and $V(X_t - X_{t+0.5}) = C(X_t - X_{t+0.5}, X_t - X_{t+0.5}) = V(X_t) - C(X_t, X_{t+0.5}) - C(X_{t+0.5}, X_t) + V(X_{t-0.5}) = 2r(0) - 2r(0.5) = 2e^0 - 2e^{-0.5} = 0.7869 \Rightarrow X_t - X_{t+0.5} \in N(0, 0.7869) \Rightarrow P(X_t - X_{t+0.05} > 1) = 1 - \Phi(\frac{1}{\sqrt{0.7869}} = 0.1292$.

(b) $r_X(\tau) = e^{-|\tau|} \Rightarrow R_X(f) = \frac{2}{1+(2\pi f)^2} \Rightarrow R_Y(f) = |H(f)|^2 R_X(f) = \left(\frac{1}{\sqrt{1+(2\pi f)^2}}\right)^2 \frac{2}{1+(2\pi f)^2}$ when $|f| < 1$ $= \left\{ \begin{array}{ll} \frac{2}{(1+(2\pi f)^2)^2} & \text{when } f \in (-1,1) \\ 0 & \text{otherwise} \end{array} \right.$
An upper bound for the variance is requested. We have by spectral decomposition that $r_Y(\tau) = \int_{\mathbb{R}} e^{i2\pi f\tau} R_Y(f)\,df$, and since $1 + (2\pi f)^2 \geq 1$ for all f we get that $V(Y_t) = r_Y(0) = \int_{\mathbb{R}} e^{i2\pi f\cdot 0} R_Y(f)\,df = \int_{-1}^{1} \frac{2}{(1+(2\pi f)^2)^2}\,df \leq \int_{-1}^{1} \frac{2}{1}\,df = 4$. □