Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
August 11, 2009, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove that if $\{X_n\}_0^N$ is a local martingale with respect to the filtration $\{\mathcal{F}_n\}_0^N$ such that $E(\min(0, X_N)) > -\infty$ and $E(\max(0, X_N)) < \infty$, then $\{X_n\}_0^N$ is a martingale with respect to $\{\mathcal{F}_n\}_0^N$. (4p)

   Solution: (See pp 100 in *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. In a financial market, each day n, the brokers buy for the amount X_n and sell for Y_n, where $X_n \in Poi(3)$ and $Y_n \in Poi(2)$ in millions of dollars and $\{X_n\}$ and $\{Y_n\}$ are indpependent between days and of each other. What is
   (a) the probability that the business volume (i.e. amount buy for and amount sell for added) exceeds \$5 million during one day? (3p)
   (b) approximately the probability that the business volume exceeds \$1.2 billion (i.e. \$1200 million) during one year (i.e. 250 days)? (5p)

   Solution:
   (a) $P(\underbrace{X_n + Y_n}_{\in Poi(5)} > 5) = 1 - P(X_n + Y_n \leq 5) = 1 - \sum_{k=0}^{5} \frac{5^k}{k!} e^{-5}$ {or from the table} $= 1 - 0.616 = 0.384$.
   (b) According to the Central Limit Theorem we have that $\sum_{k=1}^{250} X_k + \sum_{k=1}^{250} Y_k$ is approximately distributed $N(\mu, \sigma^2)$ where $\mu = E(\sum_{k=1}^{250} X_k + \sum_{k=1}^{250} Y_k) = \sum_{k=1}^{250} E(X_k) + \sum_{k=1}^{250} E(Y_k) = 250 \cdot 3 + 250 \cdot 2 = 1250$ and σ^2 is also $= 1250$ since the variables are independent and Poisson distributed. Thus $P(X_1 + Y_1 + \ldots + X_{250} + Y_{250} > 1200) \approx 1 - \Phi(\frac{1200-1250}{\sqrt{1250}}) = \Phi(1.4142) = 0.9207$. □

3. Assume $\{X_t\}$ is an AR(2) process with $E(X_t) = 100$, $V(X_t) = 4$, $C(X_t, X_{t+1}) = 3$ and $C(X_t, X_{t+2}) = 2$. Determine the parameters a_0, a_1, a_2 and σ_ϵ^2. (4p)

Solution: According to the Yule-Walker equations we get

$$\begin{cases} r(0) - a_1 r(-1) - a_2 r(-2) &= \sigma_\epsilon^2 \\ r(1) - a_1 r(0) - a_2 r(-1) &= 0 \\ r(2) - a_1 r(1) - a_2 r(0) &= 0 \end{cases} \sim \begin{cases} 4 - 3a_1 - 2a_2 &= \sigma_\epsilon^2 \quad (1) \\ 3 - 4a_1 - 3a_2 &= 0 \quad (2) \\ 2 - 3a_1 - 4a_2 &= 0 \quad (3) \end{cases}$$

$(2') : 4a_1 + 3a_2 = 3$. $(3') : 3a_1 + 4a_2 = 2$.
$3(2') - 4(3') : (9 - 16)a_2 = 9 - 8 \Rightarrow a_2 = -\frac{1}{7}$.
$(3') \Rightarrow a_1 = \frac{1}{3}(2 - (-\frac{1}{7}) \cdot 4) = \frac{6}{7}$.
$(1) \Rightarrow \sigma_\epsilon^2 = 4 - 3 \cdot \frac{6}{7} + 2 \cdot \frac{1}{7} = \frac{28-18+2}{7} = \frac{12}{7}$.
Finally, since $X_t - a_1 X_{t-1} - a_2 X_{t-2} = a_0 + \epsilon_t$ (by the definition of the AR(2) process) and since $E(X_t) = 100$ and $E(\epsilon_t) = 0$, we get $100 - a_1 100 - a_2 100 = a_0 + 0 \Rightarrow a_0 = 100 \cdot \frac{7-6+1}{7} = \frac{200}{7}$.
To summarize: $a_0 = \frac{200}{7}$, $a_1 = \frac{6}{7}$, $a_2 = -\frac{1}{7}$ and $\sigma_\epsilon^2 = \frac{12}{7}$. □

4. An index, $\{X_n\}$, is described by a GARCH(1,1) process with $a_0 = 111$, $a_1 = 0.1$ and $a_2 = 0.5$. Calculate the

   (a) variance of X_n. (4p)

   (b) probability that the index exceeds 100 at time $n+1$ given that it was 99 at time n, and that $\sigma_n = 50$. (5p)

Solution:

(a) Since $E(X_n) = 0$ and since $\sigma_n \perp \epsilon_n$ we get $\sigma^2 = E(X_n^2) = E(\sigma_n^2 \epsilon_n^2) = E(\sigma_n^2)E(\epsilon_n^2) = E(\sigma_n^2) = E(a_0 + a_1 X_{n-1}^2 + b_1 \sigma_{n-1}^2) = a_0 + a_1 E(X_{n-1}^2) + b_1 E(\sigma_n^2) \Rightarrow \sigma^2 = a_0 + a_1\sigma^2 + b_1\sigma^2 \Rightarrow \sigma^2 = \frac{a_0}{1-a_1-b_1} = \frac{111}{1-0.1-0.5} = 277.5$.

(b) $P(X_{n+1} > 100 \,|\, X_n = 99, \sigma_n = 50) = P(\sigma_{n+1}\epsilon_{n+1} > 100 \,|\, X_n = 99, \sigma_n = 50) = P(\epsilon_{n+1} > \frac{100}{\sqrt{a_0 + a_1 X_n^2 + b_1 \sigma_n^2}} \,|\, X_n = 99,\ \sigma_n = 50) = 1 - P(\epsilon_{n+1} < \frac{100}{\sqrt{111 + 0.1 \cdot 99^2 + 0.5 \cdot 50^2}}) = 1 - \Phi(\frac{100}{48.385}) = 0.0192$. □

5. Let $\{M_n\}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}$, where $\mathcal{F}_n = \sigma(M_1, M_2, \ldots, M_n)$. Show that $\{X_n\}$, defined by $X_n = e^{M_n}$, is a submartingale with respect to $\{\mathcal{F}_n\}$. (5p)

Solution: We shall prove that $E(X_{n+1} \,|\, \mathcal{F}_n) \overset{a.s.}{\geq} X_n$ for all n.
Since $f(x) = e^x$ is a convex function we have according to the Jensen inequality that $E(e^X) \geq e^{E(X)}$ for any random variable X. Thus $E(X_{n+1} \,|\, \mathcal{F}_n) = E(e^{M_{n+1}} \,|\, \mathcal{F}_n) \geq e^{E(M_{n+1} \,|\, \mathcal{F}_n)} \overset{a.s.}{=} e^{M_n} = X_n$ for all n. □