Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
January 10, 2008, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p ⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Let $\mathbb{C}_*$ be the lower price of hedging against some non-negative $\mathcal{F}_N$-measurable function f_N. Prove that if a contract is bought for a price less than $\mathbb{C}_*$, then there exists arbitrage for the buyer. (4p)

   Solution: (See pp 396–397 in *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) $\square$

2. Assume that $\{h_n : n \in \mathbb{Z}\}$ is an $AR(2)$ process with coefficients $a_0 = 1$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{4}$ and white noise variance $\sigma_\epsilon^2 = \frac{5}{16}$. Calculate

   (a) $C(h_n, h_{n+3})$. (3p)

   (b) $P(h_n - h_{n+1} < 1)$. (4p)

   Solution:

   (a) Denoting $C(h_n, h_{n+\tau})$ by R_τ we have according to the Yule-Walker equations with $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{4}$ and $\sigma_\epsilon^2 = \frac{5}{16}$ that

   $$\left\{\begin{array}{lrcl} (1): & R_0 + \frac{1}{2}R_1 - \frac{1}{4}R_2 & = & \frac{5}{16} \\ (2): & R_1 + \frac{1}{2}R_0 - \frac{1}{4}R_1 & = & 0 \\ (3): & R_2 + \frac{1}{2}R_1 - \frac{1}{4}R_0 & = & 0 \\ (4): & R_3 + \frac{1}{2}R_2 - \frac{1}{4}R_1 & = & 0 \end{array}\right. \sim \left\{\begin{array}{lrcl} (1'): & 4R_0 + 2R_1 - R_2 & = & \frac{5}{4} \\ (2'): & 2R_0 + 3R_1 & = & 0 \\ (3'): & 3R_0 + 2R_1 & = & 1 \\ (4'): & 4R_3 + 2R_2 - R_1 & = & 0 \end{array}\right.$$

   $3(3')-2(2') \Rightarrow 5R_0 = 3 \Rightarrow R_2 = \frac{1}{3}(-2\cdot\frac{3}{5}) = -\frac{2}{5} \Rightarrow R_2 - \frac{5}{4} + 4\cdot\frac{3}{5} + 2\cdot(-\frac{2}{5}) = \frac{7}{20} \Rightarrow R_3 = \frac{1}{4}(-\frac{2}{5} - 2\cdot\frac{7}{20}) = -\frac{11}{40}$. Thus $C(h_n, h_{n+3}) = -\frac{11}{40}$

   (b) Since $\{\epsilon_n\}$ is white noise with $\sigma_\epsilon^2 = \frac{5}{16}$ this means that it is a sequence of independent variables all distributed $N(0, \frac{5}{16})$. Hence $h_n - h_{n+1}$ is also normally distributed with $\mu = E(h_n - h_{n+1}) = 0$ and $\sigma^2 = D(h_n - h_{n+1}) = 2R_0 - 2R_1 = 2(\frac{3}{5} + \frac{2}{5}) = 2$. Thus $P(h_n - h_{n+1} < 1) = \Phi(\frac{1-0}{\sqrt{2}}) = \Phi(0.7071) = 0.7611$. $\square$

3. Let $\{X_n\}$ be a *simple random walk*, i.e. $X_0 = 0$ and $P(X_{n+1} = x+1 \,|\, X_n = x) = = P(X_{n+1} = x-1 \,|\, X_n = x) = \frac{1}{2}$.
   (a) Calculate $E(X_3^4)$. (3p)
   (b) Show that $\{X_n\}$ is not stationary. (4p)

Solution:

(a) $E(X_3^4) = \sum_{x_3} x^4 P(X_3 = x_3)$. After a little thinking and drawing we realize that, since $X_0 = 0$, the only possible values of X_1 are $\{-1, 1\}$, the possible values of X_2 are $\{-2, 0, 2\}$ and the possible values of $\{-3, -1, 1, 3\}$. And to the values -3 there is only one path, so $P(X_3 = -3) = (\frac{1}{2})^3 = \frac{1}{8}$. Similarly $P(X_3 = 3) = (\frac{1}{2})^3$ while -1 and 1 share the rest of the probability mass and since $P(X_3 = -1) = P(X_3 = 1)$ for symmetry reasons, we get $P(X_3 = -1) = \frac{1}{2}(1 - 2 \cdot \frac{1}{8}) = \frac{3}{8}$.
Thus $E(X_3^4) = (-3)^4 \frac{1}{8} + (-1)^4 \frac{3}{8} + 1^4 \cdot \frac{3}{8} + 3^4 \cdot \frac{1}{8} = 21$.

(b) Let $\{Z_k\}$ be a sequence of i.i.d. varibles such that $P(Z_k = -1) = P(Z_k = 1) = \frac{1}{2}$ for all $k \in \mathbb{Z}$. Then X_n can be represented as $X_n = \sum_{k=1}^n Z_k$ and
$C(X_n, X_{n+\tau}) = C\Big(\sum\limits_{k=1}^{n} Z_k, \sum\limits_{k=1}^{n+\tau} Z_k\Big) = C\Big(\sum\limits_{k=1}^{n} Z_k, \sum\limits_{k=1}^{n} Z_k\Big) + C\Big(\sum\limits_{k=1}^{n} Z_k, \sum\limits_{k=n+1}^{n+\tau} Z_k\Big) = \sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n} C(Z_j, Z_k) = \sum\limits_{k=1}^{n} C(Z_k, Z_k) = n$ (since $V(Z_k) = (-1)^2 \frac{1}{2} + 1^2 \cdot \frac{1}{2} - 0^2 = 1$) which is not a function only of the time distance τ, but rather only of the time location n. Thus $\{X_n\}$ is not stationary. □

4. Assume $\{M_n\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$. Prove that $E(e^{M_{n+1} - M_n} \,|\, \mathcal{F}_n) \geq 1$. (4p)

Solution: $E(e^{M_{n+1}-M_n} \,|\, \mathcal{F}_n) = E(e^{M_{n+1}} e^{-M_n} \,|\, \mathcal{F}_n) = e^{-M_n} E(e^{M_{n+1}} \,|\, \mathcal{F}_n) \overset{*}{\geq} \geq e^{-M_n} e^{E(M_{n+1} \,|\, \mathcal{F}_n)} = e^{-M_n} e^{M_n} = 1$ where the step $\underset{*}{\geq}$ is due to the Jensen's inequality since the exponential function is convex. □

5. Assume that $\{X_n : n \in \mathbb{Z}^+\}$ is a stochastic volatility model of order $p = 1$ with $|a_1| < 1$. Assume that $\Delta_0 \in N(\frac{a_0}{1-a_1}, \frac{c^2}{1-a_1^2})$.
   (a) Prove that $E(e^{c\delta_n}) = e^{c^2/2}$ for any $n \in \mathbb{Z}$. (4p)
   (b) Calculate $E(X_1^2)$. (4p)

Solution:

(a) Assume $Y \in N(0,1)$. Then

$$\begin{aligned} E(e^{cY}) &= \int_{\mathbb{R}} e^{cy}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{cy-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(y-c)^2/2+c^2/2}dy \\ &\overset{\{z=y-c\}}{=} e^{c^2/2}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2} \\ &= e^{c^2/2} \end{aligned}$$

(b) In the stochastic volatility model of order 1 $X_n = \sigma_n\epsilon_n$ where $\sigma_n^2 = e^{\Delta_n}$, $\Delta_n = a_0 + a_1\Delta_{n-1} + c\delta_n$ and $\{\epsilon_n\}$ and $\{\delta_n\}$ are white noise independent of each other. Then

$$\begin{aligned} E(X_n^2) &= E(\sigma_n^2)E(\epsilon_n^2) \\ &= E(e^{\Delta_n}) \\ &= E(e^{a_0+a_1\Delta_{n-1}+c\delta_n}) \\ &= e^{a_0}E(e^{a_1\Delta_{n-1}})E(e^{c\delta_n}) \end{aligned}$$

Since $\Delta_0 \in N(\frac{a_0}{1-a_1}, \frac{c^2}{1-a_1^2})$ we can write $\Delta_0 = \frac{cZ}{\sqrt{1-a_1^2}} + \frac{a_0}{1-a_1}$. Then we have that $E(e^{\Delta_0}) = \exp(\frac{a_0}{1-a_1})E\Big(\exp\Big(\frac{c}{\sqrt{1-a_1^2}}Z\Big)\Big) = \exp(\frac{a_0}{1-a_1} + \frac{c^2}{2(1-a_1^2)})$. Thus

$$\begin{aligned} E(X_1^2) &= E(e^{\Delta_1}) = E(e^{a_0+a_1\Delta_0+c\delta_1}) \\ &= e^{a_0}E(e^{a_1\Delta_0})E(e^{c\delta_1}) \\ &= e^{a_0}E\left(\exp\left(\frac{a_1c}{\sqrt{1-a_1^2}}Z + \frac{a_1a_0}{1-a_1}\right)\right)e^{c^2/2} \\ &= \exp\left(a_0 + \tfrac{c^2}{2} + \tfrac{a_0a_1}{1-a_1} + \tfrac{a_1^2c^2}{2(1-a_1^2)}\right) \end{aligned}$$

□