Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
August 11, 2008, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove the equivalence for strategies $\pi \in SF$ in a (B,S)-market with dividends

$$\left\{ \begin{array}{c} X_n^\pi = \beta_n B_n + \gamma_n(S_n + D_n) \\ B_{n-1}\Delta\beta_n + (S_{n-1} + D_{n-1})\Delta\gamma_n = 0 \end{array} \right\} \iff \Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$$

for all $n \in \mathbb{Z}^+$. (4p)

Solution: We must show that $\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$ using only that $X_n^\pi = \beta_n B_n + \gamma_n(S_n + D_n)$ and $X_n^\pi = \sum_{k=1}^n \Delta X_k$, $B_n = \sum_{k=1}^n \Delta B_k$, $S_n = \sum_{k=1}^n \Delta S_k$, $D_n = \sum_{k=1}^n \Delta D_k$ (assuming $X_0^\pi = B_0 = S_0 = D_0 = 0$).

$$\begin{aligned}
\Delta X_n^\pi &= X_n^\pi - X_{n-1}^\pi \\
&= \beta_n B_n + \gamma_n(S_n + D_n) - \beta_{n-1}B_{n-1} - \gamma_{n-1}(S_{n-1} + D_{n-1}) \\
&= \beta_n \sum_{k=1}^n \Delta B_k + \gamma_n \Big(\sum_{k=1}^n \Delta S_k + \sum_{k=1}^n \Delta D_k \Big) - \beta_{n-1} \sum_{k=1}^{n-1} \Delta B_k - \gamma_{n-1} \Big(\sum_{k=1}^{n-1} \Delta S_k + \sum_{k=1}^{n-1} \Delta D_k \Big) \\
&= (\beta_n - \beta_{n-1}) \sum_{k=1}^{n-1} \Delta B_k + (\gamma_n - \gamma_{n-1}) \Big(\sum_{k=1}^{n-1} \Delta S_k + \sum_{k=1}^{n-1} \Delta D_k \Big) + \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)
\end{aligned}$$

Here $(\beta_n - \beta_{n-1}) \sum_{k=1}^{n-1} \Delta B_k + (\gamma_n - \gamma_{n-1}) \Big(\sum_{k=1}^{n-1} \Delta S_k + \sum_{k=1}^{n-1} \Delta D_k \Big) = B_{n-1}\Delta\beta_n +$ $+ (S_{n-1} + D_{n-1})\Delta\gamma_n = 0$. Thus $\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$. □

2. Let the process $\{X_t : t \in \mathbb{Z}^+\}$ be defined by $X_t = \sum_{s=1}^{t} \epsilon_s$ where $\{\epsilon_s : s \in \mathbb{Z}^+\}$ is a sequence of independent random variables such that $P(\epsilon_s = 1) = \frac{1}{2}$ and $P(\epsilon_s = -1) = \frac{1}{2}$ for all $s \in \mathbb{Z}^+$.

   (a) What is the process $\{X_t\}$ called? (2p)

   (b) Is $\{X_t\}$ weakly stationary? (3p)

   Now let $P(\epsilon_s = 1) = 1 - P(\epsilon_s = -1) = p$ for all $s \in \mathbb{Z}^+$ where $0 < p < 1$.

   (c) Calculate the moment generating function of X_t, i.e. $m(s) = E(e^{sX_t})$. (4p)

Solution:

(a) The process $X_t = \sum_{s=1}^{t} \epsilon_s$ where $\{\epsilon_s\}$ are independent and $P(\epsilon_s = 1) = P(\epsilon_s) = \frac{1}{2}$ is commonly known as a *(simple) random walk*.

(b) To be weakly stationary $E(X_t)$ should be constant and $C(X_s, X_t)$ should be a function of $s - t$. We have that $E(X_t) = \sum_{s=1}^{t} E(\epsilon_s) = t(1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}) = 0$. Assume $s < t$. Then $C(X_s, X_t) = C(\sum_{u=1}^{s} \epsilon_u, \sum_{v=1}^{t} \epsilon_v) = C(\sum_{u=1}^{s} \epsilon_u, \sum_{v=1}^{s} \epsilon_v) + C(\sum_{u=1}^{s} \epsilon_u, \sum_{u=s+1}^{t} \epsilon_v) = \sum_{u=1}^{s}\sum_{v=1}^{s} C(\epsilon_u, \epsilon_v) + \sum_{u=1}^{s}\sum_{v=s+1}^{t} C(\epsilon_u, \epsilon_v) = \sum_{u=1}^{s} C(\epsilon_u, \epsilon_u) + 0 = s$. Assuming $s \geq t$ for the same reasons gives $C(X_s, X_t) = t$. Thus $C(X_s, X_t) = \min(s, t)$ which cannot be written as a function of $s - t$, i.e. $\{X_t\}$ is not weakly stationary.

(c) $E(e^{sX_t}) = E(e^{s(\epsilon_1+\epsilon_2+\ldots+\epsilon_t)}) = E(e^{s\epsilon_1}e^{s\epsilon_2}\cdots e^{s\epsilon_t}) = E(e^{s\epsilon_1})E(e^{s\epsilon_2})\cdots E(e^{s\epsilon_t}) = (E(e^{s\epsilon_1}))^t$ since the variables of $\{\epsilon_t\}$ are iid. Since $P(\epsilon_1 = 1) = 1 - P(\epsilon_1 = -1) = p$ we get $E(e^{s\epsilon_t}) = e^{s\cdot 1}p + e^{s\cdot(-1)}(1-p) = p(e^s - e^{-s}) + e^{-s} = 2p\sinh s + e^{-s}$. □

3. Suppose log returns $\{h_t\}$ are distributed according to the $ARCH(p)$ model. Calculate

   (a) $C(h_t, h_{t+3})$. (2p)

   (b) $D(h_t)$. (4p)

Solution:

(a) Since $E(\epsilon_t) = 0$ we have that $E(h_t) = E(\sigma_t\epsilon_t) = E(\sigma_t)E(\epsilon_t) = 0$. Thus $C(h_t, h_{t+3}) = E(h_t h_{t+3}) = E(h_t\sigma_{t+3}\epsilon_{t+3}) = E(h_t\sigma_{t+3})E(\epsilon_{t+3}) = 0$.

(b) In the $ARCH(p)$ model $h_t = \sigma_t\epsilon_t$ and $\sigma_t^2 = a_0 + \sum_{k=1}^{t} a_k h_{t-k}^2$. Therefore $D(h_t) = E(h_t^2) - (E(h_t))^2 = E(\sigma_t^2\epsilon_t^2) - 0 = E(a_0 + \sum_{k=1}^{t} a_k h_{t-k}^2)E(\epsilon_t^2) = a_0 + \sum_{k=1}^{t} a_k E(h_{t-k}^2)$. Since the $ARCH(p)$ is weakly stationary by definition, $E(h_t^2)$ is constant w.r.t. t. Thus $E(h_t^2) - \sum_{k=1}^{t} a_k E(h_{t-k}^2) = E(h_t^2)(1 - \sum_{k=1}^{t} a_k) = a_0$ so $D(h_t) = E(h_t^2) = a_0/(1 - \sum_{k=1}^{t} a_k)$. □

4. Show that the variables of the $AR(1)$ model with $a_0 = 0$ are mesokurtic. (6p)

Solution: $X_t = a_1 X_{t-1} + \sigma_\epsilon \epsilon_t$.
$E(X_t) = a_1 E(X_t) + \sigma_t E(\epsilon_t) \Rightarrow E(X_t) = 0$. $E(X_t^2) = E((a_1 X_{t-1} + \sigma_\epsilon \epsilon_t)^2) = a_1^2 E(X_{t-1}^2) + \sigma_\epsilon^2 E(\epsilon_t^2) + 2a_1\sigma_\epsilon E(X_{t-1}\epsilon_t) = a_1^2 E(X_t^2) + \sigma_\epsilon^2 \Rightarrow E(X_t^2) = \frac{\sigma_\epsilon^2}{1-a_1^2}$.
$E(X_t^4) = E((a_1 X_{t-1} + \sigma_\epsilon \epsilon_t)^4) = E(a_1^4 X_{t-1}^4 + 4a_1^3 X_{t-1}^3 \sigma_\epsilon \epsilon_t + 6a_1^2 X_{t-1}^2 \sigma_\epsilon^2 \epsilon_t^2 + +4a_1 X_{t-1}\sigma_\epsilon^3 \epsilon_t^3 + \sigma_\epsilon^4 \epsilon_t^4) = a_1^4 E(X_{t-1}^4) + 0 + 6a_1^2 E(X_{t-1}^2) e\sigma_\epsilon^2 \cdot 1 + 0 + \sigma_\epsilon^4 \cdot 3 = = a_1^4 E(X_t^4) + 6a_1^2 \sigma_\epsilon \frac{\sigma_\epsilon^2}{1-a_1^2} + 3\sigma_\epsilon^4 \Rightarrow E(X_t^4) = \frac{6a_1^2\sigma_\epsilon^4 + 3\sigma_\epsilon^4}{(1-a_1^2)(1-a_1^4)} = \frac{3\sigma_\epsilon^4(a_1^2+1)}{(1-a_1^2)(1-a_1^2)(1+a_1^2)}$.
Since $(E(X_t^2))^2 = \frac{\sigma_\epsilon^4}{(1-a_1^2)^2}$ we get that the kurtosis is
$\frac{E(X_t^4)}{(E(X_t^2))^2} - 3 = \frac{3\sigma_t^4(a_1^2+1)}{(1-a_1^2)^2(1+a_1^2)} \cdot \frac{(1-a_1^2)^2}{\sigma_\epsilon^4} - 3 = 0$. Thus the variables are mesokurtic.

□

5. Assume that $\{M_n\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$, that $E(M_n^2) < \infty$ and construct the previsible sequence $\{A_n\}$ by letting $A_0 = 0$ and $A_{n+1} = A_n + E((M_{n+1} - M_n)^2 \,|\, \mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$ Show that $\{A_n\}$ is non-decreasing. (5p)

Solution: For all $n = 0, 1, 2, \ldots$ we have that

$$\begin{aligned}
A_{n+1} &= A_n + E((M_{n+1} - M_n)^2 \,|\, \mathcal{F}_n) \\
&= A_n + E(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2 \,|\, \mathcal{F}_t) \\
&= A_n + E(M_{n+1}^2 \,|\, \mathcal{F}_t) - 2E(M_{n+1}M_n \,|\, \mathcal{F}_t) + E(M_n^2 \,|\, \mathcal{F}_t) \\
&= A_n + E(M_{n+1}^2 \,|\, \mathcal{F}_t) - 2M_n E(M_{n+1} \,|\, \mathcal{F}_t) + M_n^2 \\
&= A_n + E(M_{n+1}^2 \,|\, \mathcal{F}_t) - 2M_n^2 + M_n^2 \\
&= A_n + E(M_{n+1}^2 \,|\, \mathcal{F}_t) - (M_n)^2 \\
&= A_n + E(M_{n+1}^2 \,|\, \mathcal{F}_t) - (E(M_{n+1} \,|\, \mathcal{F}_t))^2 \\
&= A_n + D(M_{n+1} \,|\, \mathcal{F}_t) \\
&\geq A_n
\end{aligned}$$

Thus $\{A_n\}$ is non-decreasing. □