Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
August 11, 2008, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove the necessaty part of the martingale criterion of absence of arbitrage. (6p)

   Solution: (Se p 413, *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Calculate the covarinace function $R(h) = Cov(X_t, X_{t+h})$ of an $MA(1)$ process with parameters $b_0 = b_1 = \sigma_\epsilon^2 = 1$. (6p)

   Solution: $X_t = b_0 + b_1\epsilon_{t-1} + \sigma_\epsilon\epsilon_t = 1 + \epsilon_{t-1} + \epsilon_t$ where $\{\epsilon_t\}$ is white noise.
   $D(X_t) = D(1 + \epsilon_{t-1} + \epsilon_t) = 0 + D(\epsilon_{t-1}) + D(\epsilon_t) = 2.$
   $C(X_t, X_{t+1}) = C(1 + \epsilon_{t-1} + \epsilon_t\ ,\ 1 + \epsilon_t + \epsilon_{t+1}) = C(\epsilon_t, \epsilon_t) = 1.$
   $C(X_t, X_{t+h}) = C(1 + \epsilon_{t-1} + \epsilon_t\ ,\ 1 + \epsilon_{t+h-1} + \epsilon_{t+h}) = 0$ when $h \geq 2$.
   Since the covariance function is even we have that

   $$R(h) = \begin{cases} 2 & \text{if } h = 0 \\ 1 & \text{if } |h| = 1 \\ 0 & \text{if } |h| \geq 2 \end{cases}$$

   □

3. Assume that stock market log returns, $\{h_t\}$, are distributed according to the $HARCH(2)$ model with positive coefficients a_0, a_1, a_2. Show that

   $$a_1 + 2a_2 < 1$$

   (6p)

   Solution: In the $HARCH(2)$ model $h_n = \sigma_n\epsilon_n$ where $\{\epsilon_n\}$ is white noise and $\sigma_n^2 = a_0 + a_1h_{n-1}^2 + a_2(h_{n-1} + h_{n-2})^2 = a_0 + (a_1 + a_2)h_{n-1}^2 + a_2h_{n-2}^2 + 2a_2h_{n-1}h_{n-2}$. We have that $E(h_{n-1}h_{n-2}) = E(\sigma_{n-1}\epsilon_{n-1}h_{n-2}) = E(\epsilon_{n-1})E(\sigma_{n-1}h_{n-2}) = 0$ so $E(h_n^2) = E(\sigma_n^2)E(\epsilon_n^2) = E(a_0 + (a_1 + a_2)h_{n-1}^2 + a_2h_{n-2}^2 + 2a_2h_{n-1}h_{n-2}) =$ $= a_0 + (a_1 + a_2)E(h_{n-1}^2) + a_2E(h_{n-2}^2) + 2a_2E(h_{n-1}h_{n-2}) = a_0 + (a_1 + 2a_2)E(h_n^2)$, since $E(h_{n-2}^2) = E(h_{n-1}^2) = E(h_n^2)$. This means that $E(h_n^2) = \frac{a_0}{1-a_1-2a_2} > 0$ where $a_0 > 0$ and thus also $1 - a_1 - 2a_2 > 0$, i.e. $1 > a_1 + 2a_2$. □

4. Assume $\{X_n\}$ is a time homogeneous Markov chain with state space $\{0,1\}$ and transition probabilities $P(X_{n+1}=0\,|\,X_n=0)=1$ and $P(X_{n+1}=0\,|\,X_n=1)=0.01$. Show that $\{X_n\}$ is a submartingale. (6p)

Solution: $P(X_{n+1}=0\,|\,X_n=0)=1 \Rightarrow P(X_{n+1}=1\,|\,X_n=0)=0$ and $P(X_{n+1}=0\,|\,X_n=1)=0.01 \Rightarrow P(X_{n+1}=1\,|\,X_n=1)=0.99$. We get $E(X_{n+1}\,|\,X_n=0)=0\cdot P(X_{n+1}=0\,|\,X_n=0)+1\cdot P(X_{n+1}=1\,|\,X_n=0)=0=X_n$ and $E(X_{n+1}\,|\,X_n=1)=0\cdot P(X_{n+1}=0\,|\,X_n=1)+1\cdot P(X_{n+1}=1\,|\,X_n=1)=0.99\le 1=X_n$. Thus $E(X_{n+1}\,|\,X_n)\le X_n$, i.e. $\{X_n\}$ is a submartingale. □

5. Calculate $E(X_n)$ where $\{X_n : n\in\mathbb{N}\}$ is a geometric random walk[1]. (6p)

Solution: $X_n=e^{R_n}$ where $R_n=\sum_{k=1}^n U_k$ and $U_k=\begin{cases}-1 & \text{w.p. } 1/2\\ 1 & \text{w.p. } 1/2\end{cases}$.
Then, writing R_n as the linear combination $2B_n-n$ of a random variable $B_n\in Bin(n,\frac{1}{2})$ and using the binomial theorem, we have that

$$\begin{aligned}
E(X_n) &= E(e^{R_n})\\
&= \sum_{j=-n,-n+2,\ldots,n-2,n} e^j P(R_n=j)\\
&= \sum_{j=,-n+2,\ldots,n-2,n} e^j P(2B_n-n=j)\\
&= \sum_{j=-n,-n+2,\ldots,n-2,n} e^j P(B_n=\tfrac{1}{2}(j+n))\\
&= \sum_{j=0}^{n} e^{2j-n}P(B_n=j)\\
&= \sum_{j=0}^{n} e^{2j-n}\tbinom{n}{j}(\tfrac{1}{2})^j(\tfrac{1}{2})^{n-j}\\
&= \sum_{j=0}^{n} (e^2)^j e^{-n}\tbinom{n}{j}\tfrac{1}{2^n}\\
&= (2e)^{-n}\sum_{j=0}^{n} e^{-n}\tbinom{n}{j}(e^2)^j\cdot 1^{n-j}\\
&= (2e)^{-n}(e^2+1)^n\\
&= \left(\frac{e^2+1}{2e}\right)^n.
\end{aligned}$$

Alternatively, $E(X_n)=E(e^{\sum_{k=1}^n U_k})=(E(e^{U_1}))^n=(e^{-1}\cdot\frac{1}{2}+e^{1}\cdot\frac{1}{2})^n=(\frac{1+e^2}{2e})^n$ (which is another way of writing $\cosh^n(1)$). □

[1] A *geometric random walk* is a random process, $\{X_n\}$, satisfying $X_n=e^{R_n}$ where the process $\{R_n\}$ is a random walk