Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
October 24, 2007, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

For each problem a *complete* solution should be given. All solutions should be thoroughly presented. Each solution should start at the top of a new sheet of paper. Only one solution a sheet.
The proper solutions will be available on the internet at `http://www.hh.se/staff/erja` → `Teaching` → `Financial Mathematics` → `Stochastic models` → `Previous exams` → `071024: Solution`

1. Show that a local martingale is a generalised martingale, i.e. if $X = \{X_n, \mathcal{F}_n : n \in \mathbb{Z}^+\}$ is a random process with $E(|X_0|) < \infty$, then you should prove that $X \in \mathcal{M}_{loc} \Rightarrow X \in G\mathcal{M}$, (where $\mathcal{M}_{loc}$ is the class of local martingales and $G\mathcal{M}$ is the class of generalised martingels). (4p)

2. Show that the variables of a process of the GARCH family are uncorrelated.(3p)

3. Suppose $\{h_t : t \in \mathbb{Z}^+\}$ is a random walk with $h_0 = 0$ and

$$h_{t+1} = h_t + \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -2 & \text{with probability } \frac{1}{2} \end{cases}$$

   (a) Calculate the covariance function $R(s,t) = Cov(h_s, h_t)$. (4p)

   (b) Determine the deterministic function $C(t)$ such that the process $\{h_t + C(t) : t \in \mathbb{Z}^+\}$ is a martingale with respect to the flow $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma(h_0, h_1, \ldots, h_t)$. (3p)

4. Let $\{X_t : t \in \mathbb{Z}^+\}$ be an $AR(1)$ process with white noise variance σ_ϵ^2 and with parameters $a_0 = 0$ and $a_1 = a$. Assuming $\{X_t\}$ is stationary, calculate

   (a) the second moment of X_t, (4p)

   (b) the fourth moment of X_t, (5p)

   (c) the white noise variance σ_ϵ^2 if $a = 0.9$ and $D(X_t) = 1$. (3p)

5. Consider a (B,S)-market model $(\{B_n\}, \{S_n\})$ where $B_0 > 0$, $S_0 > 0$, $B_{n+1} = (1+r)B_n$, $S_{n+1} = (1+\rho)S_n$ for all n and $r > -1$, $\rho > -1$. Show that $E(\rho) \geq r$ implies that the process $\{\frac{S_n}{B_n} : n \in \mathbb{Z}^+\}$ is a submartingale with respect to the flow $\mathcal{F}_n = \sigma(\frac{S_0}{B_0}, \frac{S_1}{B_1}, \ldots, \frac{S_n}{B_n})$. (4p)

GOOD LUCK!