Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
October 24, 2007, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show that a local martingale is a generalised martingale, i.e. if $X = \{X_n, \mathcal{F}_n : n \in \mathbb{Z}^+\}$ is a random process with $E(|X_0|) < \infty$, then you should prove that $X \in \mathcal{M}_{loc} \;\Rightarrow\; X \in G\mathcal{M}$, (where $\mathcal{M}_{loc}$ is the class of local martingales and $G\mathcal{M}$ is the class of generalised martingels). (4p)

   Solution: (Se p 98, *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Show that the variables of a process of the GARCH family are uncorrelated.(3p)

   Solution: In the processes $\{h_n\}$ of the GARCH family $h_n = \sigma_n \epsilon_n$ for all n where ϵ_n is white noise and $m \leq n \Rightarrow \sigma_m \perp \epsilon_n$. Thus $E(h_n) = E(\sigma_n)E(\epsilon_n) = E(\sigma_n)\cdot 0 = 0$. Now assume $m < n$. Then $E(h_m h_n) = E(h_m \sigma_n \epsilon_n) = E(h_m \sigma_n)E(\epsilon_n) = E(h_m \sigma_n) \cdot 0 = 0$. Since $m \neq n \;\Rightarrow\;$ either $m < n$ or $m > n$ we have that $E(h_m h_n) = 0$ for all $m \neq n$ so $Cov(h_m, h_n) = 0$ for all $m \neq n$, which is to say that the variables $\{h_n\}$ are uncorrelated. □

3. Suppose $\{h_t : t \in \mathbb{Z}^+\}$ is a random walk with $h_0 = 0$ and

$$h_{t+1} \;=\; h_t + \begin{cases} \;\;1 & \text{with probability } \frac{1}{2} \\ -2 & \text{with probability } \frac{1}{2} \end{cases}$$

   (a) Calculate the covariance function $R(s,t) = Cov(h_s, h_t)$. (4p)

   (b) Determine the deterministic function $C(t)$ such that the process $\{h_t + C(t) : t \in \mathbb{Z}^+\}$ is a martingale with respect to the flow $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma(h_0, h_1, \ldots, h_t)$. (3p)

Solution:

(a) Let $\{Z_t\}$ be a sequence of independent variables with $Z_t = \begin{cases} 1 & \text{w.p. } 1/2 \\ -2 & \text{w.p. } 1/2 \end{cases}$.
Then $h_t = h_{t-1} + Z_t = h_{t-2} + Z_{t-1} + Z_t = \ldots = h_s + Z_{s+1} + \ldots + Z_t$ and since $\{Z_t\}$ are iid we have that $h_t - h_s = Z_{s+1} + \ldots + Z_t \perp Z_1 + \ldots + Z_s = h_s$, so assuming $s < t$ we have that $R(s,t) = Cov(h_s, h_t) = Cov(h_s, h_t - h_s + h_s) = Cov(h_s, h_t - h_s) + Cov(h_s, h_s) = 0 + D(h_s)$. In the same way, for $s \geq t$ we have $R(s,t) = \ldots = 0 + D(h_t)$. This means that in general $R(s,t) = D(h_{\min(s,t)})$. Now, $D(h_s) = D(Z_1 + \ldots + Z_s) \overset{iid}{=} s\,D(Z_1)$ where $D(Z_1) = E(Z_1^2) - E(Z_1)^2 = 1^2 \cdot \frac{1}{2} + (-2)^2 \cdot \frac{1}{2} - (1 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{2})^2 = \frac{9}{4}$ so $D(h_s) = \frac{9}{4}s$ and $R(s,t) = D(h_{\min(s,t)}) = \frac{9}{4}\min(s,t)$.

(b) We shall find $C(t)$ such that $\{h_t + C(t)\}$ is a martingale.
$E(h_{t+1} \,|\, \mathcal{F}_t) = E(h_t + Z_{t+1} \,|\, \mathcal{F}_t) = h_t + E(Z_{t+1}) = h_t + 1 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{2} = h_t + \frac{1}{2}$.
Now, let $g_t = h_t + C(t)$. Then $E(g_{t+1} \,|\, \mathcal{F}_t) = E(h_t + Z_{t+1} + C(t+1) \,|\, \mathcal{F}_t) = h_t + E(Z_{t+1}) + C(t+1) = h_t - \frac{1}{2} + C(t+1)$. This equals $g_t = h_t + C(t)$ if $-\frac{1}{2} + C(t+1) = C(t)$, i.e. if $C(t+1) - C(t) = \frac{1}{2}$ for all t.
To deduce that $C(t) = \frac{1}{2}t + a$, where $a \in \mathbb{R}$ one can reason as follows.
<u>Alternative I</u>: Differentiate w.r.t. t both sides of $C(t+1) - C(t) = \frac{1}{2}$ $(*)$. This yields $C'(t+1) = C'(t)$ for all $t \Rightarrow C'(t) = b$ for some $b \in \mathbb{R} \Rightarrow C(t) = a + bt$ where $a, b \in \mathbb{R}$. Now $(*) \Rightarrow b = \frac{1}{2}$, so $C(t) = a + \frac{1}{2}t$.
<u>Alternative II</u>: Since we are dealing with discrete time one may use the difference equation approach. $C(t) = C(t-1) + \frac{1}{2} = C(t-2) + \frac{1}{2} + \frac{1}{2} = \ldots = C(0) + t \cdot \frac{1}{2}$. Thus $C(t) = a + \frac{1}{2}t$ where $a = C(0)$ is some real number.□

4. Let $\{X_t : t \in \mathbb{Z}^+\}$ be an $AR(1)$ process with white noise variance σ_ϵ^2 and with parameters $a_0 = 0$ and $a_1 = a$. Assuming $\{X_t\}$ is stationary, calculate

(a) the second moment of X_t, (4p)

(b) the fourth moment of X_t, (5p)

(c) the white noise variance σ_ϵ^2 if $a = 0.9$ and $D(X_t) = 1$. (3p)

Solution:

(a) $X_t = aX_{t-1} + \sigma_\epsilon \epsilon_t$ where $\{\epsilon_t\}$ are independent and $\epsilon_t \in N(0,1)$.
$m_1 = E(X_t) = aE(X_{t-1}) + \sigma_\epsilon E(\epsilon_t) \overset{stationarity}{\Rightarrow} m_1(1-a) = 0 \Rightarrow E(X_t) = m_1 = 0$.
$m_2 = E(X_t^2) = E(a^2 X_{t-1}^2 + 2a\sigma_\epsilon X_{t-1}\epsilon_t + \sigma_\epsilon^2 \epsilon_t^2) =$
$a^2 E(X_{t-1}^2) + 2a\sigma_\epsilon E(X_{t-1})E(\epsilon_t) + \sigma_\epsilon^2 E(\sigma_t^2) \overset{stationarity}{\Rightarrow}$
$m_2(1-a^2) = 0 + \sigma_\epsilon^2 \cdot 1 \Rightarrow E(X_t^2) = m_2 = \frac{\sigma_\epsilon^2}{1-a^2}$.

(b) $m_4 = E(X_t^4) = E((a^2X_{t-1}^2 + 2a\sigma_\epsilon X_{t-1}\epsilon_t + \sigma_\epsilon^2\epsilon_t^2)^2) =$
$E(a^4X_{t-1}^4 + 4a^2\sigma_\epsilon^2 X_{t-1}^2\epsilon_t^2 + \sigma_\epsilon^4\epsilon_t^4 + 4a^3\sigma_\epsilon X_{t-1}^3\epsilon_t + 2a^2\sigma_\epsilon^2 X_{t-1}^2\epsilon_t^2 + 4a\sigma_\epsilon^3 X_{t-1}\epsilon_t^3) =$
$= a^4E(X_{t-1}^4) + 4a^2\sigma_\epsilon^2 E(X_{t-1}^2)E(\epsilon_t^2) + \sigma_\epsilon^4 E(\epsilon_t^4) + 4a^3\sigma_\epsilon E(X_{t-1}^3)E(\epsilon_t) +$
$+2a^2\sigma_\epsilon^2 E(X_{t-1}^2)E(\epsilon_t^2) + 4a\sigma_\epsilon^3 E(X_{t-1})E(\epsilon_t^3) =$
$= a^4 m_4 + 4a^2\sigma_\epsilon^2 \frac{\sigma_\epsilon^2}{1-a^2} + 3\sigma_\epsilon^4 + 0 + 2a^2\sigma_\epsilon^2\frac{\sigma_\epsilon^2}{1-a^2} + 0$
(since $\{X_t\}$ is stationary and $E(\epsilon_t) = E(X_t) = 0$)
$\Rightarrow\ m_4(1-a^4) = \dfrac{6a^2\sigma_\epsilon^4}{1-a^2} + 3\sigma_\epsilon^4$
$\Rightarrow\ E(X_t^4) = m_4 =$

$$= \frac{6a^2\sigma_\epsilon^4 + 3\sigma_\epsilon^4(1-a^2)}{(1-a^2)(1-a^4)} = \frac{3\sigma_\epsilon^4(1+a^2)}{(1-a^2)(1-a^2)(1+a^2)} = \frac{3\sigma_\epsilon^4}{(1-a^2)^2}.$$

(c) $X_t = aX_{t-1} + \sigma_\epsilon\epsilon_t$
$1 = D(X_t) = D(aX_{t-1} + \sigma_\epsilon\epsilon_t) = a^2D(X_{t-1}) + \sigma_\epsilon^2 D(\epsilon_t) = a^2 + \sigma_\epsilon^2$. $a = 0.9 \Rightarrow$
$\sigma_\epsilon^2 = 1 - 0.9^2 = 0.19$. □

5. Consider a (B,S)-market model $(\{B_n\},\{S_n\})$ where $B_0 > 0$, $S_0 > 0$, $B_{n+1} = (1+r)B_n$, $S_{n+1} = (1+\rho)S_n$ for all n and $r > -1$, $\rho > -1$. Show that $E(\rho) \geq r$ implies that the process $\{\frac{S_n}{B_n} : n \in \mathbb{Z}^+\}$ is a submartingale with respect to the flow $\mathcal{F}_n = \sigma(\frac{S_0}{B_0}, \frac{S_1}{B_1}, \ldots, \frac{S_n}{B_n})$. (4p)

Solution: Assume $E(\rho) \geq r$. Then

$$\begin{aligned} E\left(\frac{S_{n+1}}{B_{n+1}}\,\middle|\,\mathcal{F}_n\right) &= E\left(\frac{(1+\rho)S_n}{(1+r)B_n}\,\middle|\,\mathcal{F}_n\right) \\ &= \frac{1+E(\rho)}{1+r}\cdot\frac{S_n}{B_n} \\ &\geq \frac{1+r}{1+r}\cdot\frac{S_n}{B_n} \\ &= \frac{S_n}{B_n} \end{aligned}$$

$\Rightarrow\ \{\frac{S_n}{B_n}\}$ is a submartingale. □