Solutions to Exam for Stochastic models in Discrete time
2.5 points/3.75 ECTS

Master's program of Financial Mathematics
January 2, 2007, 13.30 – 17.30

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show that if $\{X_n : n \geq 0\}$ is a local martingale with respect to the filtration $\{\mathcal{F}_n\}$ and $\max(E(X_n^+), E(X_n^-)) < \infty$ (where $X_n^+ = \max(0, X_n)$ and $X_n^- = -\min(0, X_n)$), then $\{X_n : n \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. (5p)

   Solution: (Se p 100, *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda = 3$.

   (a) Calculate $P(N_5 > 4)$. (3p)

   Is the process $\{Y_n : n \in \mathbb{Z}^+\}$ defined by $Y_n = \sum_{k=1}^{n}(-1)^k(N_k - N_{k-1})$ for all $n = 1, 2, 3, \ldots$

   (b) weakly stationary? (4p)

   (c) a martingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$? (4p)

   Solution:

   (a) $P(N_5 > 4) = 1 - P(N_5 \leq 4) = 1 - P(N_5 = 4) - P(N_5 = 3) - P(N_5 = 2) - P(N_5 = 1) - P(N_5 = 0) = 1 - \frac{(3\cdot5)^4}{4!}e^{-3\cdot5} - \frac{15^3}{3!}e^{-15} - \frac{15^2}{2!}e^{-15} - \frac{15^1}{1!}e^{-15} - \frac{15^0}{0!}e^{-15} = 1 - 2800.375e^{-15} = 0.9991434$. □

   (b) $Y_n = -(N_1 - N_0) + (N_2 - N_1) - (N_3 - N_2) + \ldots + (-1)^n(N_n - N_{n-1})$. Thus $E(Y_1) = E(-(N_1 - N_0) = -(E(N_1) - 0) = -3$ and $E(Y_2) = = E(-(N_1 - N_0) + (N_2 - N_1)) = E(-2N_1 + N_2) = -2E(N_1) + E(N_2) = 0$. But for weak stationarity $E(Y_n)$ should be constant with respect to n. Thus $\{Y_n\}$ is not weakly stationary. □

   (c) $E(Y_{n+1}|\mathcal{F}_n) = E(\sum_{k=1}^{n+1}(-1)^k(N_k - N_{k-1})|Y_1, \ldots, Y_n) = E((-1)^{n+1}(N_{n+1} - N_n) + Y_n|Y_1, \ldots, Y_n) = E((-1)^{n+1}(N_{n+1} - N_n)) + Y_n = (-1)^{n+1}(3(n+1) - 3n) + Y_n = 3(-1)^{n+1} + Y_n$. But $(-1)^{n+1} \neq 0$ for all $n = 1, 2, 3, \ldots$ so $\{Y_n\}$ cannot be a martingale. □

3. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(2)$ process with $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{3}$ and $\sigma_\epsilon^2 = \frac{1}{6}$ with covariance function $R(h) = C(X_t, X_{t+h})$. Determine $R(0)$, $R(1)$ and $R(2)$. (5p)

Solution: For simplicity let us denot $R(h)$ by R_h, $h \in \mathbb{Z}^+$ and remember that $R(-k) = R(k)$. According to the Yule-Walker equations we have that

$$\left\{\begin{array}{lcl} R_0 - (a_1R_1 + a_2R_2) & = & \sigma_\epsilon^2 \\ R_1 - (a_1R_0 + a_2R_1) & = & 0 \\ R_2 - (a_1R_1 + a_2R_0) & = & 0 \end{array}\right. \sim \left\{\begin{array}{lcl} R_0 - \frac{1}{2}R_1 + \frac{1}{3}R_2 & = & \frac{1}{6} \\ R_1 - \frac{1}{2}R_0 + \frac{1}{3}R_1 & = & 0 \\ R_2 - \frac{1}{2}R_1 + \frac{1}{3}R_0 & = & 0 \end{array}\right. \sim$$

$$\sim \left\{\begin{array}{lcl} 6R_0 - 3R_1 + 2R_2 & = & 1 \\ -3R_0 + 8R_1 & = & 0 \\ 2R_0 - 3R_1 + 6R_2 & = & 0 \end{array}\right. \sim \left\{\begin{array}{lcl} 6R_0 - 3R_1 + 2R_2 & = & 1 \\ -3R_0 + 8R_1 & = & 0 \\ 16R_0 - 6R_1 & = & 3 \end{array}\right. \sim$$

$$\sim \left\{\begin{array}{lcl} 6R_0 - 3R_1 + 2R_2 & = & 1 \\ -3R_0 + 8R_1 & = & 0 \\ (-18 + 128)R_0 & = & 24 \end{array}\right. \sim \left\{\begin{array}{lcl} R_0 = \frac{12}{55} & = & 0.2181818 \\ R_1 = -\frac{3-16R_0}{6} = \frac{9}{110} & = & 0.0818182 \\ R_2 = \frac{1}{2}(1 - 6R_0 + 3R_1) = -\frac{7}{220} & = & -0.0318182 \end{array}\right.$$

□

4. Suppose the process $\{h_n\}$ is distributed according to the volatility model of first order (i.e. with $p = 1$). Show that the process variables $\{h_n\}$ are uncorrelated.(4p)

Solution: In the volatility model of order 1 we have that (according to the summary of formulae) $h_n = \sigma_n\epsilon_n$ where $\sigma_n^2 = e^{\Delta_n}$ and $\Delta_n = a_0 + a_1\Delta_{n-1} + c\delta_n$ and where $\{\epsilon_n\}$ and $\{\delta_n\}$ are white noise processes indpendent of each other. Since $E(\epsilon_n) = 0$ and $\epsilon_n \perp \sigma_n$ we have for $k > 0$ that $C(h_n, h_{n+k}) = E(h_nh_{n+k}) - E(h_n)E(h_{n+k}) = E(h_n\sigma_{n+k}\epsilon_{n+k}) - E(\sigma_n\epsilon_n)E(\sigma_{n+k}\epsilon_{n+k}) = E(h_n\sigma_{n+k})\underbrace{E(\epsilon_{n+k})}_{=0} - E(\sigma_n)\underbrace{E(\epsilon_n)}_{=0}E(\sigma_{n+k})\underbrace{E(\epsilon_{n+k})}_{=0} = 0$, i.e. the variables $\{h_n\}$ are uncorrelated. □

5. Show that if $\{X_n\}$ and $\{Y_n\}$ are supermartingales with respect to the filtrations $\{\mathcal{F}_n^X\}$ and $\{\mathcal{F}_n^Y\}$ respectively, then $\{\min(X_n, Y_n)\}$ is also a supermartingale with respect to the filtration $\{\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y)\}$. (5p)

Hint: Remember Jensens inequality: $g : \mathbb{R}^n \to \mathbb{R}$ concave $\Rightarrow$ $E(g(\boldsymbol{X})) \leq g(E(\boldsymbol{X}))$.

Solution: $E(X_{n+1}|\mathcal{F}_n^X) \leq X_n$, $E(Y_{n+1}|\mathcal{F}_n^Y) \leq Y_n$. Then because $\min(x, y)$ is a concave function in x and y we have according to Jensens inequality that $E(\min(X_{n+1}, Y_{n+1}|\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y)) \leq \min(\underbrace{E(X_{n+1}|\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y))}_{\leq X_n}, \underbrace{E(Y_{n+1}|\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y))}_{\leq Y_n}) \leq$ $\leq \min(X_n, Y_n)$. Thus $\{\min(X_n, Y_n)\}$ is a supermartingale with respect to $\{\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y)\}$. □