Solutions to Exam for Stochastic models in Discrete time 2.5 points/3.75 ECTS

Master's program of Financial Mathematics
May 4, 2007, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show that if $\mathbb{C}^*$ is the upper price of hedging against some non-negative $\mathcal{F}_N$-measurable function f_N and a contract is sold for a price greater than $\mathbb{C}^*$, then there exists an opportunity for arbitrage for the seller. (5p)

   Solution: (See pp 396–397 in *Essentials of stochastic finance. Facts, models, theory* by A.N. Shiryaev.) □

2. Suppose $\{h_t : t \in \mathbb{Z}^+\}$ is an $AR(1)$ process with parameters $a_1 = 0.78$ and white noise variance $\sigma_\epsilon^2 = 1$. Then calculate

   (a) $V(h_t)$, (3p)

   (b) $P(h_t + h_{t-1} \leq 1)$. (4p)

   Solution:

   (a) Since $h_s \perp \epsilon_t$ whenever $s < t$ we have that $\sigma_h^2 = V(h_t) = V(0.78h_{t-1} + \epsilon_t) = = 0.78^2 V(h_{t-1}) + \sigma_\epsilon^2 = 0.78^2\sigma_h^2 + 1 \Rightarrow \sigma_h^2(1 - 0.78^2) = 1 \Rightarrow \sigma_h^2 = 2.5536.$

   (b) All variables of the AR process are normally distributed so in order to calculate the probability $P(h_t + h_{t-1} \leq 1)$ we need just to find $\mu = E(h_t + h_{t-1})$ and $\sigma^2 = (h_t + h_{t-1})$. We get $E(h_t + h_{t-1}) = E(h_t) + E(h_{t-1}) = 0$ and $V(h_t + h_{t-1}) = C(h_t + h_{t-1}, h_t + h_{t-1}) = V(h_t) + V(h_{t-1}) + 2C(h_t, h_{t-1}) = 2\sigma_h^2 + 2R(1)$. From the Yule Walker equations we have that $k = 0 : R(0) - 0.78R(-1) = 1$ and $k = 1 : R(1) - 0.78R(0) = 0$. Since $R(-1) = R(1)$ we get $R(0) = V(h_t) = 2.5536$ and $R(1) = C(h_t, h_{t-1}) = 2.5536$. Therefore $V(h_t + h_{t-1}) = 2 \cdot 2.5536 + 2 \cdot 1.9918 = 9.0909$. Thus $P(h_t + h_{t+1} \leq 1) = = \Phi(\frac{1-0}{\sqrt{9.0909}}) = 0.6293$. □

3. Suppose that $\{h_t : t \in \mathbb{Z}^+\}$ is a $GARCH(1,1)$ model.
   (a) Calculate the first moment of $\{h_t\}$. (3p)
   (b) Caclulate the second moment of $\{h_t\}$. (4p)
   (c) Assuming $a_1 > 0$, show that $\{h_t\}$ is leptokurtic. (6p)

Solution:

(a) $E(h_t) = E(\sigma_t \epsilon_t) = E(\sigma_t) \cdot 0 = 0$.

(b) $E(h_t^2) = E(\sigma_t^2)E(\epsilon_t^2) = E(a_0 + a_1 h_{t-1}^2 + b_1 \sigma_{t-1}^2) = a_0 + a_1 E(h_t^2) + b_1 E(\sigma_{t-1}^2)$. Here $E(\sigma_{t-1}^2) = E(\sigma_{t-1}^2 \epsilon_{t-1}^2) = E(h_{t-1}^2) = E(h_t^2)$ so $E(h_t^2)(1 - a_1 - b_1) =$ $= a_0 \Rightarrow E(h_t^2) = \frac{a_0}{1-a_1-b_1}$.

(c) To show that $\{h_t\}$ is leptokurtic we need to show that $\frac{E(h_t^4)}{(E(h_t^2))^2} - 3 > 0$. $E(h_t^4) = \frac{3a_0^2(1+a_1+b_1)}{(1-a_1-b_1)(1-3a_1^2-b_1^2-2a_1b_1)}$ where by the definition of the GARCH model $a_0 > 0$ and $a_1 \geq 0$, $b_1 \geq 0$. Thus the kurtosis is $\frac{E(h_t^4)}{(E(h_t^2))^2} - 3 =$ $= \frac{3a_0^2(1+a_1+b_1)}{(1-a_1-b_1)(1-3a_1^2-b_1^2-2a_1b_1)} \cdot \frac{(1-a_1-b_1)^2}{a_0^2} - 3 = \frac{6a_1^2}{1-3a_1^2-b_1^2-2a_1b_1}$ and this is > 0 iff $a_1 > 0$ and $1-3a_1^2-b_1^2-2a_1b_1 > 0$. Now, $a_1 > 0$, $b_1 \geq 0$ and $E(h_{t-1}^4) \geq 0 \Rightarrow$

$$\Rightarrow \begin{cases} (1 - a_1 - b_1 > 0 \text{ and } 1 - 3a_1^2 - b_1^2 - 2a_1b_1 > 0) \\ \text{or} \\ (1 - a_1 - b_1 < 0 \text{ and } 1 - 3a_1^2 - b_1^2 - 2a_1b_1 < 0) \end{cases}$$

However, since $E(h_t^2) = \frac{a_0}{1-a_1-b_1} \geq 0$ and $a_0 > 0$ we have that $1 - a_1 - b_1 > 0$ implying that $1 - 3a_1^2 - b_1^2 - 2a_1b_1 > 0$ which means that the kurtosis is positive, i.e. that the process $\{h_t\}$ is leptokurtic. □

4. Is the process $\{X_n : n = 0, 1, 2, \ldots\}$ a martingale, a submartingale or a supermartingale with respect to the σ-algebra $\mathcal{F}_n = \{\sigma(Y_0, Y_1, \ldots, Y_n)\}$ where $X_n = \max(-1, Y_n)$ and $\{Y_n\}$ is a random walk with $Y_0 = 0$ and $P(Y_n - Y_{n-1} = 1) =$ $= P(Y_n - Y_{n-1} = -2) = 0.5$ for all $n = 1, 2, 3, \ldots$? (5p)

Solution: Conditional on $Y_{n-1} = y_{n-1}$ we have that

$$X_n = \begin{cases} \max(-1, 1 + y_{n-1}) & \text{with probability } 0.5 \\ \max(-1, -2 + y_{n-1}) & \text{with probability } 0.5 \end{cases}.$$

Thus $E(X_n | Y_{n-1} = y_{n-1}) = 0.5\max(-1, 1 + y_{n-1}) + 0.5\max(-1, -2 + y_{n-1}) =$ $= \max(-0.5 + (-0.5), 0.5y_{n-1} + 0.5 + 0.5y_{n-1} - 0.5 \cdot 2) = \max(-1, y_{n-1} - 0.5) \leq$ $\leq \max(-1, y_{n-1})$. Thus $\{X_n\}$ is a supermartingale. □