Solutions to Exercise Exam for Stochastic models in Discrete time
2.5 points/3.75 ECTS

Master's program of Financial Mathematics
October 20, 2006, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show the equivalence for strategies $\pi \in SF$ in a (B,S)-market with dividends

$$\left\{ \begin{array}{c} X_n^\pi = \beta_n B_n + \gamma_n(S_n + D_n) \\ B_{n-1}\Delta\beta_n + (S_{n-1} + D_{n-1})\Delta\gamma_n = 0 \end{array} \right\} \iff \Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$$

for all $n \in \mathbb{Z}^+$. (3p)

Solution: We must show that $\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$ using only that $X_n^\pi = \beta_n B_n + \gamma_n(S_n + D_n)$ and $X_n^\pi = \sum_{k=1}^n \Delta X_k$, $B_n = \sum_{k=1}^n \Delta B_k$, $S_n = \sum_{k=1}^n \Delta S_k$, $D_n = \sum_{k=1}^n \Delta D_k$ (assuming $X_0^\pi = B_0 = S_0 = D_0 = 0$).

$$\begin{aligned}
\Delta X_n^\pi &= X_n^\pi - X_{n-1}^\pi \\
&= \beta_n B_n + \gamma_n(S_n + D_n) - \beta_{n-1}B_{n-1} - \gamma_{n-1}(S_{n-1} + D_{n-1}) \\
&= \beta_n \sum^n \Delta B_k + \gamma_n \Big(\sum^n \Delta S_k + \sum^n \Delta D_k \Big) - \beta_{n-1} \sum^{n-1} \Delta B_k - \gamma_{n-1} \Big(\sum^{n-1} \Delta S_k + \sum^{n-1} \Delta D_k \Big) \\
&= (\beta_n - \beta_{n-1}) \sum^{n-1} \Delta B_k + (\gamma_n - \gamma_{n-1}) \Big(\sum^{n-1} \Delta S_k + \sum^{n-1} \Delta D_k \Big) + \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)
\end{aligned}$$

(compare with (9) on page 386 in *Essentials of Stochastic Finance* by A.N. Shiryaev). Here $(\beta_n - \beta_{n-1}) \sum^{n-1} \Delta B_k + (\gamma_n - \gamma_{n-1}) \Big(\sum^{n-1} \Delta S_k + \sum^{n-1} \Delta D_k \Big) =$ $= B_{n-1}\Delta\beta_n + (S_{n-1} + D_{n-1})\Delta\gamma_n = 0$. Thus $\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \Delta D_n)$. (This proves the statement in both directions of implication.) □

2. Calculate the second moment of the stationary HARCH(2) process. (3p)

Solution: $h_n = \sigma_n \epsilon_n$ where $\{\epsilon_n\}$ is white noise and
$\sigma_n^2 = a_0 + (a_1 + a_2)h_{n-1}^2 + a_2 h_{n-2}^2 + 2a_2 h_{n-1}h_{n-2}$.
Since $E(h_{n-1}h_{n-2}) = E(\epsilon_{n-1}\sigma_{n-1}h_{n-2}) = \underbrace{E(\epsilon_{n-1})}_{=0} E(\sigma_{n-1}h_{n-2}) = 0$

we have that the second moment is

$$E(h_n^2) = E(\sigma_n^2)\underbrace{E(\epsilon_n^2)}_{=1} = E(a_0 + (a_1 + a_2)h_{n-1}^2 + a_2 h_{n-2}^2 + 2a_2 h_{n-1}h_{n-2}) =$$

$$= a_0 + (a_1 + a_2)\underbrace{E(h_{n-1}^2)}_{=E(h_n^2)} + a_2 \underbrace{E(h_{n-2}^2)}_{=E(h_n^2)} + 2a_2 \underbrace{E(h_{n-1}h_{n-2})}_{=0} \Rightarrow$$

$$\Rightarrow E(h_n^2)(1 - (a_1 + a_2) - a_2) = a_0 \Rightarrow E(h_n^2) = \frac{a_0}{1 - a_1 - 2a_2}. \qquad \square$$

3. Let $\{X_n : n \in \mathbb{Z}^+\}$ be a random walk white noise increments $\{\epsilon_n\}$.
   (a) Is the process $\{Y_n\}$ defined by $Y_n = \frac{1}{\sqrt{n}} X_n$ weakly stationary? (4p)
   (b) Show that $E(X_n) = 0$ but $\forall C \in \mathbb{R} : \lim_{n\to\infty} P(|X_n| > C) = 1$. (5p)

Solution:

(a) $Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \epsilon_k$ where $\epsilon_k \in N(0,1)$ and $i \neq j \Rightarrow \epsilon_i \perp \epsilon_j$.
For $\{Y_k\}$ to be weakly stationary we must have that $E(Y_n) = m$ and $Cov(Y_n, Y_{n+h}) = R(h)$. We do have that $E(Y_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} E(\epsilon_k) = 0$ and $Cov(Y_n, Y_n) = D(Y_n) = \frac{1}{n} \sum_{k=1}^{n} D(\epsilon_k)$ {because ϵ_k are independent} $= 1$ independently of n. But

$$\begin{aligned} Cov(Y_n, Y_{n+h}) &= Cov\Big(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \epsilon_k, \frac{1}{\sqrt{n+h}} \sum_{k=1}^{n+h} \epsilon_k\Big) \\ &= \frac{1}{\sqrt{n(n+h)}} \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} Cov(\epsilon_i, \epsilon_j) + \sum_{i=1}^{n} \sum_{j=n+1}^{n+h} Cov(\underbrace{\epsilon_i, \epsilon_j}_{i\neq j})\Big) \\ &= \frac{1}{\sqrt{n(n+h)}} \sum_{i=1}^{n} D(\epsilon_i) + 0 \\ &= \sqrt{\frac{n}{n+h}} \end{aligned}$$

which is *not* independent of n. Thus $\{Y_n\}$ is *not* weakly stationary.

(b) $E(\epsilon_k)=0 \;\Rightarrow\; E(X_n)=\sum\limits_{k=1}^{n} E(\epsilon_k)=0$ and

$\{D(\epsilon_k)=1$ and $\epsilon_i \perp \epsilon_j$ whenever $i\neq j\} \;\Rightarrow\; D(X_n)=\sum\limits_{k=1}^{n} D(\epsilon_k)=n$

$\epsilon_k \in N(0,1) \;\Rightarrow\; X_n \in N(0,n)$

$\Rightarrow\;\; P(|X_n|>C) = P\Big(\{X_n<-C\}\cup\{X_n>C\}\Big) = 2P(X_n>C) =$

$= 2\Big(1-\Phi(\frac{C}{\sqrt{n}})\Big)$. Now $\lim\limits_{n\to\infty} P(|X_n|>C) = \lim\limits_{n\to\infty} 2\Big(1-\Phi(\frac{C}{\sqrt{n}})\Big) =$

$= 2\Big(1-\lim\limits_{n\to\infty}\Phi(\frac{C}{\sqrt{n}})\Big) = 2\Big(1-\lim\limits_{n\to\infty}\int_{-\infty}^{C/\sqrt{n}}\Phi(dx)\Big) = 2(1-\Phi(0)) = 1.$ □

4. Consider the following game with a single dice: in round n we throw a fair dice and if the dice shows
   - 6, then we get 6 SEK
   - 1, 2, 3 or 4, then we must pay that amount SEK
   - 5, then we throw the dice again and then if the dice shows
     - 5 or 6, then we get twice that amount SEK
     - 1, 2, 3 or 4 then we must pay that amount SEK

   Determine whether this game is a martingale, a supermartingale or a submartingale for us. (3p)

Solution: Let g_n be our gain from round n. Then the total gain by round n is $G_n=\sum_{k=1}^{n} g_k$ and the question is whether $\{G_n\}$ is a martingale, a supermartingale or a submartingale.

Let D_1 be the outcome of the first throw and D_2 the outcome of the (possible) second throw. Then we have that

$P(g_n=6)=P(D_1=6)=\frac{1}{6}$

$P(g_n=2\cdot 5)=P(D_1=5\,\cap\,D_2=5)=\frac{1}{36}$

$P(g_n=2\cdot 6)=P(D_1=5\,\cap\,D_2=6)=\frac{1}{36}$

$P(g_n=-1)=P(\{D_1=1\}\cup\{D_1=5\,\cap\,D_2=1\})=\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{6}=\frac{7}{36}$

$P(g_n=-2)=\ldots=\frac{7}{36}=P(g_n=-3)=P(g_n=-4)$

Now, let $\Omega_g=\{-4,-3,-2,-1,6,10,12\}$. Then $E(g_n)=$

$=\sum\limits_{k\in\Omega_g} kP(g_n=k) = -1\cdot\frac{7}{36}-2\cdot\frac{7}{36}-3\cdot\frac{7}{36}-4\cdot\frac{7}{36}+6\cdot\frac{1}{6}+10\cdot\frac{1}{36}+12\cdot\frac{7}{36}=-\frac{1}{3}.$

With $\mathcal{F}_n=\sigma(g_1,\ldots,g_n)$ we have that $E(G_{n+1}\,|\,\mathcal{F}_n)=E(G_n+g_{n+1}\,|\,g_1,\ldots,g_n)=$

$=G_n+E(g_{n+1})=G_n-\frac{1}{3}\leq G_n$. Thus $\{G_n\}$ is a supermartingale. □

5. Let $\{X_t\}$ be a stationary $AR(2)$ process with coefficients $a_1 = 0$ and $a_2 = a \neq 0$.

(a) Calculate the value of σ_ϵ^2 if $a = \frac{1}{2}$ and $D(X_t) = 1$. (3p)

(b) Determine the covariance function of $\{X_t\}$. (4p)

(c) Derive the maximum likelihood estimator of a. (5p)

Solution:

(a) $X_t = \frac{1}{2}X_{t-2} + \sigma_\epsilon \epsilon_t$ for all $t \Rightarrow$
$1 = D(X_t) = D(\frac{1}{2}X_{t-2} + \sigma_\epsilon \epsilon_t) = \frac{1}{4}D(X_{t-2}) + \sigma_\epsilon^2 D(\epsilon_t) = \frac{1}{4} + \sigma_\epsilon^2 \Rightarrow \sigma_\epsilon^2 = \frac{3}{4}$.

(b) $R(1) = Cov(X_t, X_{t-1}) = Cov(aX_{t-2} + \sigma_\epsilon \epsilon_t, X_{t-1}) = aCov(X_{t-2}, X_{t-1}) + 0 =$
$= aR(1) \Rightarrow R(1) = 0$
$R(2) = Cov(X_t, X_{t-2}) = Cov(aX_{t-2} + \sigma_\epsilon \epsilon_t, X_{t-2}) = aD(X_{t-2}) + 0 = a$
$R(3) = 0$
$R(4) = Cov(X_t, X_{t-4}) = Cov(aX_{t-2} + \sigma_\epsilon \epsilon_t, X_{t-4}) = aCov(X_{t-2}, X_{t-4}) + 0 =$
$= aR(2) = a^2$
$\vdots$

$$R(h) = \begin{cases} a^{h/2} & \text{if } h \text{ is even} \\ 0 & \text{o.w.} \end{cases}$$

(c) $E(X_t \,|\, X_{t-2}) = aX_{t-2}$ and $D(X_t \,|\, X_{t-2}) = \sigma_\epsilon^2$ so the density funciton of x_t given $x_{t-1}, x_{t-2}, \ldots$ is $f(x_t \,|\, x_{t-1}, x_{t-2}, \ldots; a) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp(-\frac{1}{2\sigma_\epsilon^2}(x_t - ax_{t-2})^2)$ and so the likelihood function, $L(a)$ (i.e. the joint density of the sample $\boldsymbol{x}_t = (x_1, x_2, \ldots, x_t)$ conditional on the two first observations, x_1 and x_2) is $f(\boldsymbol{x}_t, \,|\, x_1, x_2) = \prod_{t=3}^{n} f(x_t \,|\, x_{t-2}; a) = \prod_{t=3}^{n} \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp(-\frac{1}{2\sigma_\epsilon^2}(x_t - ax_{t-2})^2) = (2\pi\sigma_\epsilon^2)^{-(n-2)/2} \exp\Big(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=3}^{n}(x_t - ax_{t-2})^2\Big)$. The log likelihood is therefore

$\ell(a) = \ln L(a) = -\frac{n-2}{2}\ln(2\pi\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2}\sum_{t=3}^{n}(x_t - ax_{t-2})^2$.

To maximise this we want to solve the equation

$$\frac{d\ell}{da}(a) = 0 \Rightarrow -\frac{1}{2\sigma_\epsilon^2}\sum_{t=3}^{n} 2(-x_{t-2})(x_t - ax_{t-2}) = 0$$
$$\Rightarrow \sum_{t=3}^{n} x_t x_{t-2} = \sum_{t=3}^{n} a x_{t-2}^2$$
$$\Rightarrow \hat{a} = \frac{\sum_{t=3}^{n} x_t x_{t-2}}{\sum_{t=3}^{n} x_{t-2}^2}$$

□