Exam for Stochastic models in Discrete time
2.5 points/3.75 ECTS

Master's program of Financial Mathematics
January 2, 2007, 13.30 – 17.30

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

For each problem a *complete* solution should be given. All solutions should be thoroughly presented. Each solution should start at the top of a new sheet of paper. Only one solution a sheet.

The proper solutions will be available on the internet at
`http://www.hh.se/staff/erja` → `Teaching` → `Financial Mathematics` → `Stochastic models` → `Previous exams` → `070102: Solution`

1. Show that if $\{X_n : n \geq 0\}$ is a local martingale with respect to the filtration $\{\mathcal{F}_n\}$ and $\max(E(X_n^+), E(X_n^-)) < \infty$ (where $X_n^+ = \max(0, X_n)$ and $X_n^- = -\min(0, X_n)$), then $\{X_n : n \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. (5p)

2. Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda = 3$.
   (a) Calculate $P(N_5 > 4)$. (3p)

   Is the process $\{Y_n : n \in \mathbb{Z}^+\}$ defined by $Y_n = \sum_{k=1}^{n} (-1)^k (N_k - N_{k-1})$ for all $n = 1, 2, 3, \ldots$

   (b) weakly stationary? (4p)
   (c) a martingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$? (4p)

3. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(2)$ process with $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{3}$ and $\sigma_\epsilon^2 = \frac{1}{6}$ with covariance function $R(h) = C(X_t, X_{t+h})$. Determine $R(0)$, $R(1)$ and $R(2)$. (5p)

4. Suppose the process $\{h_n\}$ is distributed according to the volatility model of first order (i.e. with $p = 1$). Show that the process variables $\{h_n\}$ are uncorrelated. (4p)

5. Show that if $\{X_n\}$ and $\{Y_n\}$ are supermartingales with respect to the filtrations $\{\mathcal{F}_n^X\}$ and $\{\mathcal{F}_n^Y\}$ respectively, then $\{\min(X_n, Y_n)\}$ is also a supermartingale with respect to the filtration $\{\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y)\}$. (5p)

   Hint: Remember Jensens inequality: $g : \mathbb{R}^n \to \mathbb{R}$ concave $\Rightarrow$ $E(g(\boldsymbol{X})) \leq g(E(\boldsymbol{X}))$.

GOOD LUCK!