

SOLUTIONS TO EXAM FOR STOCHASTIC MODELS IN DISCRETE TIME
2.5 POINTS/3.75 ECTS

Master's program of Financial Mathematics
October 25, 2006, 9.00 – 13.00

Max number of points: 30.

Halmstad University grading bounds: 12p \Rightarrow grade 3, 18p \Rightarrow grade 4, 24p \Rightarrow grade 5.

ECTS bounds: 12p \Rightarrow grade E, 15p \Rightarrow grade D, 18p \Rightarrow grade C, 21p \Rightarrow grade B, 24p \Rightarrow grade A.

Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.

Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

For each problem a *complete* solution should be given. All solutions should be thoroughly presented. Each solution should start at the top of a new sheet of paper. Only one solution a sheet.

The proper solutions will be available on the internet at

<http://www.hh.se/staff/erja> \rightarrow Teaching \rightarrow Financial Mathematics \rightarrow Stochastic models \rightarrow Previous exams \rightarrow 061025: Solution

1. Formulate and prove the necessity part of the Martingale Criterion of the Absence of Arbitrage. (6p)

Solution: (See *Essentials of Stochastic Finance* by A.N. Shiryaev, p 417.) \square

2. Let $X_n = \sum_{k=0}^n c_k \epsilon_k$ where $c_k \in \mathbb{R}$ for all k and $\{\epsilon_k\}$ is white noise. Then what kind of process is $\{X_n\}$ and is it weakly stationary if

(a) $k_0 = 0$ and $c_k = 1$ for all k ? (4p)

(b) $k_0 = n - 2$ and $c_k = 2^{k-n}$? (4p)

Solution:

(a) $X_n = \sum_{k=0}^n \epsilon_k$ where $i \neq j \Rightarrow \epsilon_i \perp \epsilon_j$ and $E(\epsilon_k) = 0 \Rightarrow \{X_n\}$ is a random walk. This process is not weakly stationary because $Cov(X_n, X_{n+h}) =$
 $= Cov\left(\sum_{k=0}^n \epsilon_k, \sum_{k=0}^{n+h} \epsilon_k\right) = Cov\left(\sum_{k=0}^n \epsilon_k, \sum_{k=0}^n \epsilon_k\right) + Cov\left(\sum_{k=0}^n \epsilon_k, \sum_{k=n+1}^{n+h} \epsilon_k\right) =$
 $= \sum_{i=0}^n \sum_{j=0}^n \underbrace{Cov(\epsilon_i, \epsilon_j)}_{=1 \text{ if } i=j, 0 \text{ o.w.}} + \sum_{i=0}^n \sum_{j=n+1}^{n+h} \underbrace{Cov(\epsilon_i, \epsilon_j)}_{\perp} = \sum_{i=0}^n 1 + 0 = n + 1$ which apparently is a function of n .

- (b) $X_n = \sum_{k=n-2}^n 2^{k-n} \epsilon_k = 2^{-2} \epsilon_{n-2} + 2^{-1} \epsilon_{n-1} + 2^0 \epsilon_n = \frac{1}{4} \epsilon_{n-2} + \frac{1}{2} \epsilon_{n-1} + \epsilon_n \Rightarrow \{X_n\}$
 is an $MA(2)$ process which is weakly stationary with covariance function

$$R(h) = \begin{cases} \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2 = \frac{21}{16} & \text{if } h = 0 \\ \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{5}{8} & \text{if } |h| = 1 \\ \frac{1}{4} \cdot 1 = \frac{1}{4} & \text{if } |h| = 2 \\ 0 & \text{if } |h| \geq 3 \end{cases} \quad \square$$

3. Let $\{X_t\}$ be a stationary $ARCH(1)$ process.

(a) Calculate the variance of X_t . (4p)

(b) Show that $\{X_t\}$ is leptokurtic. (4p)

Solution:

(a) $X_t = \sigma_t \epsilon_t$ where $\{\epsilon_t\}$ is white noise and $\sigma_t^2 = a_0 + a_1 X_{t-1}^2$

$$E(X_t) = E(\underbrace{\sigma_t \epsilon_t}_{\perp}) = E(\sigma_t) \underbrace{E(\epsilon_t)}_{=0} = 0$$

$$E(X_t^2) = E(\underbrace{\sigma_t^2 \epsilon_t^2}_{\perp}) = E(\sigma_t^2) \underbrace{E(\epsilon_t^2)}_{=1} = E(a_0 + a_1 X_{t-1}^2) = a_0 + a_1 \underbrace{E(X_{t-1}^2)}_{E(X_t^2)} \Rightarrow$$

$$\Rightarrow E(X_t^2)(1 - a_1) = a_0 \Rightarrow E(X_t^2) = \frac{a_0}{1 - a_1} \Rightarrow$$

$$\Rightarrow D(X_t) = E(X_t^2) - \underbrace{E(X_t)^2}_{=0} = \frac{a_0}{1 - a_1}.$$

(b) $E(X_t^4) = E(\sigma_t^4) \underbrace{E(\epsilon_t^4)}_{=3} = 3E((a_0 + a_1 X_{t-1}^2)^2) =$

$$= 3(a_0^2 + 2a_0 a_1 \underbrace{E(X_{t-1}^2)}_{=3} + a_1^2 E(X_{t-1}^4)) = 3a_0^2 + 6a_0 a_1 \cdot \frac{a_0}{1 - a_1} + 3a_1^2 E(X_{t-1}^4) \Rightarrow$$

$$\Rightarrow E(X_t^4)(1 - 3a_1^2) = \frac{3a_0^2(1 - a_1) + 6a_0^2 a_1}{1 - a_1} \Rightarrow E(X_t^4) = \frac{3a_0^2(1 + a_1)}{(1 - 3a_1^2)(1 - a_1)}.$$

Thus to be stationary $|a_1| < \frac{1}{\sqrt{3}}$. Regarding the kurtosis we have that

$$E(X_t) = 0 \Rightarrow \text{Kurtosis} = \frac{E(X_t^4)}{(E(X_t^2))^2} - 3 = \frac{a_0^2(1 + a_1)}{(1 - 3a_1^2)(1 - a_1)} \cdot \frac{(1 - a_1)^2}{a_0^2} - 3 = \frac{6a_1^2}{1 - 3a_1^2} > 0$$

(since $|a_1| < \frac{1}{\sqrt{3}}$) which proves that $\{X_t\}$ is leptokurtic. □

4. Each day n an agent considers

- either buying one asset for B_n SEK and, immediately, selling it for S_n SEK
- or do nothing and wait for next day.

The agent has the strategy to buy and sell at day n if $S_n > B_n$ and wait otherwise, which gives him the gain G_n of earnings up to day n . Assume that the sequences $\{B_n\}$ and $\{S_n\}$ are independent at all levels and that $B_n \in N(101, 1)$ and $S_n \in N(100, 1)$.

- (a) Show that the gain, $\{G_n\}$, from this strategy is a submartingale. (4p)
- (b) To come to terms with this arbitrage, a fee paid by the agent at each time n should be determined so that $\{G_n\}$ is a martingale. How large should this fee be? (4p)

Solution:

- (a) The earnings from day n is $S_n - B_n$ if the agent buys an asset at day n or 0 if he does not. Thus $G_n = G_{n-1} + (S_n - B_n)I(S_n > B_n) = G_{n-1} + (S_n - B_n)I(S_n - B_n > 0)$. Now, letting $Z = S_n - B_n$ we have that $E(S_n - B_n | S_n > B_n) = E(Z | Z > 0) = \int_0^\infty z F_Z(dz) \geq 0$. since $z \in [0, \infty) \Rightarrow z \geq 0$ and $F_Z(dz) = f_Z(z)dz \geq 0$. Thus (with $\mathcal{F}_n = \sigma(G_1, \dots, G_n)$) $E(G_n | \mathcal{F}_{n-1}) = E(G_{n-1} + (S_n - B_n)I(S_n > B_n) | G_{n-1}, \dots, G_1) = G_{n-1} + E(Z | Z > 0) \geq G_{n-1}$ so $\{G_n\}$ is a submartingale.

- (b) We have that $E(S_n - B_n) = 100 - 101 = -1$. Since S_n and B_n are independent $D(S_n - B_n) = D(S_n) + D(B_n) = 2$. Thus, since S_n and B_n are normally distributed $S_n - B_n \in N(-1, 2)$. Now, letting $Z = S_n - B_n$ we have that $E(S_n - B_n | S_n > B_n) = E(Z | Z > 0) = \int_0^\infty z F_Z(dz) = \int_0^\infty z d\Phi\left(\frac{z+1}{\sqrt{2}}\right) = \int_0^\infty z \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left(-\frac{1}{2 \cdot 2}(z+1)^2\right) dz \left\{ \begin{array}{ll} y = \frac{1}{2}(z+1) & z \in (0, \infty) \\ dy = \frac{1}{2} dz & y \in (\frac{1}{2}, \infty) \end{array} \right\} = \int_{1/2}^\infty \frac{2y-1}{2\sqrt{\pi}} e^{-y^2/2} 2dy = \frac{2}{\sqrt{\pi}} \underbrace{\int_{1/2}^\infty y e^{-y^2/2} dy}_I - \sqrt{2} \underbrace{\int_{1/2}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy}_{=1-\Phi(0.5)}$. Here
- $$I = \left\{ \begin{array}{ll} u = e^{-y^2/2} & y \in (\frac{1}{2}, \infty) \\ du = -ye^{-y^2/2} dy & u \in (e^{-1/8}, 0) \end{array} \right\} \int_{e^{-1/8}}^0 (-du) = e^{-1/8} \Rightarrow$$
- $$\Rightarrow E(S_n - B_n | S_n > B_n) = \frac{2}{\sqrt{\pi}} e^{-1/8} - \sqrt{2}(1 - \Phi(0.5)) = 0.5595062.$$
- So the fee should be 0.5595062 SEK. \square