13. In the $ARMA(1,1)$ model calculate the mean and variance under stationarity.

14. In the $AR(2)$ model derive the MLE of the parameters $\alpha_0, \alpha_1, \alpha_2$. The selling price of the share *Ericsson B* at *Stockholms fondbörs* during 2006-09-25 had the following record:

Hour	*Minute*	*Price*	*Hour*	*Minute*	*Price*
8	00	25.2	12	00	25.2
	15	25.3		15	25.25
	30	25.3		30	25.2
	45	25.4		45	25.25
9	00	25.5	13	00	25.25
	15	25.35		15	25.2
	30	25.45		30	25.2
	45	25.35		45	25.25
10	00	25.25	14	00	25.25
	15	25.35		15	25.2
	30	25.35		30	25.25
	45	25.3		45	25.15
11	00	25.35	15	00	25.2
	15	25.25		15	25.25
	30	25.25		30	25.2
	45	25.3		45	25.15

Source: Dagens industri 2006-09-25, `http://www.di.se`

Assuming an $AR(2)$ model for these data and estimate the expected value by first estimating the AR parameters. (The solution should be presented together with program code.)

15. In the $ARCH(1)$ model, assuming stationarity,
    (a) calculate the fourth moment,
    (b) which are the feasible values of the coefficients?
    (c) show that the distribution is *leptokurtic*, Assume that the observation $\{x_0, x_1, \ldots, x_n\}$ of an $ARCH(1)$ model is made.
    (d) Show that, conditional on $X_0 = x_0$, the Maximum likelikhood estimators of the parameters a_0 and a_1 are the values of these parameters which satisfy the equations

$$\begin{cases} \sum_{k=1}^{n} \frac{1}{a_0 + a_1 x_{k-1}^2} & = \frac{1}{2} \sum_{k=1}^{n} \frac{x_k^2}{(a_0 + a_1 x_{k-1}^2)^2} \\ \sum_{k=1}^{n} \frac{x_{k-1}^2}{a_0 + a_1 x_{k-1}^2} & = \frac{1}{2} \sum_{k=1}^{n} \frac{x_k^2 x_{k-1}^2}{(a_0 + a_1 x_{k-1}^2)^2} \end{cases}$$

13. ARMA(1,1) : $X_t = a_0 + a_1 X_{t-1} + b_1 \varepsilon_{t-1} + \sigma_\varepsilon \varepsilon_t$

Then

$$E(X_t) = E(a_0 + a_1 X_{t-1} + b_1 \varepsilon_{t-1} + \sigma_\varepsilon \varepsilon_t)$$
$$= a_0 + a_1 \underbrace{E(X_{t-1})}_{= E(X_t)} + 0 + 0$$

$$\Rightarrow (1-a_1)E(X_t) = a_0 \quad \Rightarrow \quad E(X_t) = \frac{a_0}{1-a_1}$$

$$V = D(X_t)$$
$$= D(a_0 + a_1 X_{t-1} + b_1 \varepsilon_{t-1} + \sigma_\varepsilon \varepsilon_t)$$
$$= D(a_1 X_{t-1} + b_1 \varepsilon_{t-1}) + D(\sigma_\varepsilon \varepsilon_t)$$
$$= a_1^2 \underbrace{D(X_{t-1})}_{= D(X_t) = V} + b_1^2 \underbrace{D(\varepsilon_{t-1})}_{=1} + 2a_1 b_1 C(X_{t-1}, \varepsilon_{t-1}) + \sigma_\varepsilon^2 \underbrace{D(\varepsilon_t)}_{=1}$$
$$= a_1^2 V + b_1^2 + \sigma_\varepsilon^2 + 2a_1 b_1 C(a_0 + a_1 X_{t-2} + b_1 \varepsilon_{t-2} + \sigma_\varepsilon \varepsilon_{t-1}, \ \varepsilon_{t-1})$$
$$= a_1^2 V + b_1^2 + \sigma_\varepsilon^2 + 2a_1 b_1 (0 + 0 + 0 + \sigma_\varepsilon)$$

$$\Rightarrow (1-a_1^2) V = b_1^2 + \sigma_\varepsilon^2 + 2a_1 b_1 \sigma_\varepsilon$$

$$\Rightarrow D(X_t) = \frac{b_1^2 + \sigma_\varepsilon^2 + 2a_1 b_1 \sigma_\varepsilon}{1-a_1^2}$$

$$\left(\text{If } \sigma_\varepsilon = 1 \text{ we get } D(X_t) = \frac{b_1^2 + 2a_1 b_1 + 1}{1-a_1^2}\right)$$

14 AR(2)

$$X_t = a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \sigma_\varepsilon \varepsilon_t$$

To calculate MLE's we want to maximize the likelihood function

$$f(X; a_0, a_1, a_2, \sigma_\varepsilon \mid X_1, X_2) =$$

$$= \prod_{t=3}^{N} \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} e^{-\frac{1}{2\sigma_\varepsilon^2}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})^2}$$

$$\doteq (2\pi\sigma_\varepsilon^2)^{-N/2} e^{-\frac{1}{2\sigma_\varepsilon^2}\sum_{t=3}^{N}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})^2}$$

Maximizing this is equivalent to maximizing the log

$$l(a_0, a_1, a_2, \sigma_\varepsilon) = C - N \ln \sigma_\varepsilon - \frac{1}{2\sigma_\varepsilon^2}\sum_{t=3}^{N}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})^2$$

Differentiating wrt the different parameters we get

$$\frac{\partial l}{\partial a_0} = +\frac{1}{\sigma_\varepsilon^2}\sum_{t=3}^{N}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})$$

$$\frac{\partial l}{\partial a_1} = \frac{1}{\sigma_\varepsilon^2}\sum_{t=3}^{N} X_{t-1}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})$$

$$\frac{\partial l}{\partial a_2} = \frac{1}{\sigma_\varepsilon^2}\sum_{t=3}^{N} X_{t-2}(X_t - a_0 - a_1 X_{t-1} - a_2 X_{t-2})$$

To maximize we calculate a_0, a_1, a_2 such that $\frac{\partial \ell}{\partial a_0} = \frac{\partial \ell}{\partial a_1} = \frac{\partial \ell}{\partial a_2} = 0$. We get

$$\begin{cases} \underbrace{\sum X_t}_{S1} = a_0 (N-2) + a_1 \underbrace{\sum X_{t-1}}_{S2} + a_2 \underbrace{\sum X_{t-2}}_{S3} \\ \underbrace{\sum X_t X_{t-1}}_{S4} = a_0 \sum X_{t-1} + a_1 \underbrace{\sum X_{t-1}^2}_{S5} + a_2 \underbrace{\sum X_{t-1} X_{t-2}}_{S6} \\ \underbrace{\sum X_t X_{t-2}}_{S7} = a_0 \sum X_{t-2} + a_1 \sum X_{t-1} X_{t-2} + a_2 \underbrace{\sum X_{t-2}^2}_{S8} \end{cases}$$

In the example we get

$$758.2 = 30 a_0 + 758.35 a_1 + 758.35 a_2$$
$$19166.15 = 758.35 a_0 + 19170.01 a_1 + 19169.93 a_2$$
$$19166.14 = 758.35 a_0 + 19169.93 a_1 + 19170.01 a_2$$

$$\begin{cases} \hat{a}_0 = 5.322 \\ \hat{a}_1 = 0.488 \\ \hat{a}_2 = 0.301 \end{cases}$$

$$v = V(X_t) = V(a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \sigma_\varepsilon \varepsilon_t)$$
$$= a_1^2 v + a_2^2 v + 2 a_1 a_2 r_X(1) + \sigma_\varepsilon^2$$
$$v = \frac{2 a_1 a_2 r_X(1) + \sigma_\varepsilon^2}{1 - a_1^2 - a_2^2}$$

$$m = E(X_t) = E(a_0 + a_1 X_{t-1} + a_2 X_{t-2} + \sigma_\varepsilon \varepsilon_t)$$
$$= a_0 + a_1 m + a_2 m \Rightarrow E(X_t) = \frac{a_0}{1 - a_1 - a_2}$$
$$\Rightarrow E(X_t) = 25.22275$$

Exercise II.13

```
# In the statistics program R one may write as follows.

# Input the data
h <- c(25.20,25.30,25.30,25.40,25.50,25.35,25.45,25.35,25.25,
     25.35,25.35,25.30,25.35,25.25,25.25,25.30,25.20,25.25,
     25.20,25.25,25.25,25.20,25.20,25.25,25.25,25.20,25.25,
     25.15,25.20,25.25,25.20,25.15)
# The cofficients are
s1 <- 30
s2 <- sum(h[2:31])
s3 <- sum(h[1:30])
s4 <- s2
s5 <- sum(h[2:31]^2)
s6 <- sum(h[2:31]*h[1:30])
s7 <- s3
s8 <- s6
s9 <- sum(h[1:30]^2)
# and the values of the vector
s10 <- sum(h[3:32])
s11 <- sum(h[3:32]*h[2:31])
s12 <- sum(h[3:32]*h[1:30])
# Then the MLEs are the elements of
# inv(A) * b where
# A is the matrix
A <- matrix(c(s1,s2,s3,s4,s5,s6,s7,s8,s9),3,3)
# and b is the vector
b <- c(s10,s11,s12)
# so the MLEs are
solve(A) %*% b
# => a0 = 5.8219655
#    a1 = 0.4815181
#    a2 = 0.2879697
# which gives us the estimate of the AR(2) process varaibles
a0/(1-a1-a2)
# => 25.25665

# In R one may check this by using the
# standard function arima which takes
# a vector of data (in our case h) and
# a vector of parameters (p,d,q) (in our case (2,0,0))
arima(h,c(2,0,0))
# This tells us that the estimate of the AR coefficients
# are a0 = 6.907954, a1 = 0.4394 and a2 = 0.2871
# (and the sigma^2 is estimated as 0.003794)
```

15(a) ARCH(1)

$h_t = \sigma_t \varepsilon_t$ where $\{\varepsilon_t\}$ is white noise and $\sigma_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2$.

For calculating the fourth moment we first look at the second

$$E(h_t^2) = E(\underset{\perp}{\sigma_t^2 \varepsilon_t^2}) = E(\sigma_t^2)\underbrace{E(\varepsilon_t^2)}_{=1} =$$

$$= E(\alpha_0 + \alpha_1 h_{t-1}^2) = \alpha_0 + \alpha_1 \underbrace{E(h_{t-1}^2)}_{=E(h_t^2) \text{ assuming stationarity}}$$

$$\Rightarrow E(h_t^2) = \frac{\alpha_0}{1-\alpha_1}.$$ Now

$$E(h_t^4) = E(\sigma_t^4)\underbrace{E(\varepsilon_t^4)}_{=3} \Rightarrow E(\sigma_t^4) = \frac{1}{3}E(h_t^4)$$

and $E(h_t^4) = 3E((\alpha_0 + \alpha_1 h_{t-1}^2)^2) =$

$$= 3E(\alpha_0^2 + 2\alpha_0\alpha_1 h_{t-1}^2 + \alpha_1^2 h_{t-1}^4)$$

$$= 3\alpha_0^2 + 6\alpha_0\alpha_1 \underbrace{E(h_{t-1}^2)}_{\frac{\alpha_0}{1-\alpha_1}} + 3\alpha_1^2 \underbrace{E(h_{t-1}^4)}_{=E(h_t^4) \text{ under stronger stationarity}}$$

$$\Rightarrow E(h_t^4) = \frac{3\alpha_0^2 + \frac{6\alpha_0^2\alpha_1}{1-\alpha_1}}{1-3\alpha_1^2} =$$

$$= \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)}$$

(b) $$E(h_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)}$$

where (by definition) $\alpha_0 > 0$, $\alpha_1 \geq 0$

Then $1+\alpha_1 > 0$ and since $E(h_t^4) > 0$ we have that $(1-3\alpha_1^2)(1-\alpha_1) > 0$.

Further since $0 < E(h_t^2) = \frac{\alpha_0}{1-\alpha_1}$

we also have that $0 \leq \alpha_1 < 1$

and thus $1-3\alpha_1^2 > 0$ which means that $0 \leq \alpha_1 < \frac{1}{\sqrt{3}}$. Thus feasible parameter values are

$$\left\{(\alpha_0, \alpha_1) : \alpha_0 > 0,\ 0 \leq \alpha_1 < \frac{1}{\sqrt{3}}\right\}$$

(c) $$\text{Kurtosis} = \frac{E(h_t^4)}{(E(h_t^2))^2} - 3 =$$

$$= \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)} \cdot \frac{(1-\alpha_1)^2}{\alpha_0^2} - 3$$

$$= \frac{3(1+\alpha_1)(1-\alpha_1)}{1-3\alpha_1^2} - \frac{3(1-3\alpha_1^2)}{1-3\alpha_1^2}$$

$$= \frac{3(1-\alpha_1^2-1+3\alpha_1^2)}{1-3\alpha_1^2} = \frac{3 \cdot 2\alpha_1^2}{1-3\alpha_1^2} > 0$$

$\Rightarrow$ the process variables are leptokurtic.