Solutions to Exam for Random Processes, 7.5 ECTS

January 17, 2009, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume that $\{X_t\}$ is a weakly stationary random process which is differentiable in squared mean, and that r_X is the cvf of $\{X_t\}$ and $r_{X'}$ is the cvf of $\{X'_t\}$. Prove that $r_{X'}(\tau) = -r''_X(\tau)$. (4p)

   Solution: (See p 100 in the course literature *Random processes*.) □

2. Assume that $\{\epsilon_t : t \in \mathbb{Z}\}$ is white noise, and that the process $\{X_t : t \in \mathbb{Z}\}$ is defined by $X_t = \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$ for all $t \in \mathbb{Z}$.

   (a) What is the process $\{X_t\}$ called? (3p)

   (b) Prove that $\{X_t\}$ is weakly stationary. (3p)

   Solution:

   (a) An $MA(q)$ process $\{Y_t\}$ is defined by the equation $Y_t = \sum_{k=1}^{q} c_k\epsilon_{t-k} + \sigma_\epsilon\epsilon_t$ where $\{\epsilon_t\}$ is white noise. Thus $\{X_t\}$ in this case is an $MA(2)$ process with parameters $c_1 = -2$, $c_2 = 1$ and $\sigma_\epsilon = 1$.

   (b) Since $\{\epsilon_t\}$ is white noise, $E(\epsilon_t) = 0$ and $C(\epsilon_s, \epsilon_t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$. Thus $m(t) = E(\epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}) = 0 - 2 \cdot 0 + 0 = 0$. $C(X_s, X_t)$ is considered in the special cases when $s = t$, $s = t + 1$, $s = t + 2$, ... When $s = t$ we have $C(X_s, X_t) = V(\epsilon_t) + V(-2\epsilon_{t-1}) + V(\epsilon_{t-2}) = 1 + (-2)^2 \cdot 1 + 1 = 6$. When $s = t+1$ we have $C(X_s, X_t) = C(\epsilon_{t+1} - 2\epsilon_t + \epsilon_{t-1}, \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}) = C(-2\epsilon_t, \epsilon_t) + C(\epsilon_{t-1}, -2\epsilon_{t-1}) = -4$. When $s = t + 2$ we have $C(X_s, X_t) = C(\epsilon_{t+2} - 2\epsilon_{t+1} + \epsilon_t, \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}) = C(\epsilon_t, \epsilon_t) = 1$. And finally for $s \geq t+3$ we have that $C(X_s, X_t) = C(\epsilon_s - 2\epsilon_{s-1} + \epsilon_{s-2}, \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}) = 0$ since no indexes are alike. For $s = t - 1$ we get $C(X_s, X_t) = -4$ in a calculation similar to when $s = t + 1$, when $s = t - 2$ we get $C(X_s, X_t) = 1$ in a calculation similar to when $s = t + 2$, and when $s \leq t - 3$ we get $C(X_s, X_t) = 0$ in a calculation similar to when $s \geq t + 3$. To sum up we have totally that the expectation function is $m(t) = 0$ not a funciton of t, and the covariance fuction is $r(s,t) = \begin{cases} 6 & \text{if } s - t = 0 \\ -4 & \text{if } |s-t| = 1 \\ 1 & \text{if } |s-t| = 2 \\ 0 & \text{if } |s-t| \geq 3 \end{cases}$ which is just a function of the time distance, $|s - t|$. Thus $\{X_t\}$ is weakly stationary. □

3. Consider the weakly stationary random process $\{X_t : t \in \mathbb{R}\}$ with expectation function $m(t) = 1$ and spectral density function $R(f) = f^2 e^{-|f|}$.

   (a) Determine the covariance function of $\{X_t\}$. (3p)

   Assume the random process $\{X_t\}$ is filtered with impulse response $h(t) = \delta_{-1}(t) + 2\delta_0(t) + \delta_1(t)$.

   (b) Calculate the spectral density of the filtered signal. (3p)

   Solution:

   (a) According to the tables we have that the spectral density of a process with cvf $|\tau|^k e^{-\alpha|\tau|}$ is $R(f) = \frac{k!\big((\alpha+i2\pi f)^{k+1}+(\alpha-i2\pi f)^{k+1}\big)}{(\alpha^2+(2\pi f)^2)^{k+1}}$. Using the theorem about inversion of the Fourier transform and the symmetry of the covariance function we thus have that the covariance function in this case with $\alpha = 1$ and $k = 2$ is $r(\tau) = \frac{2((1+i2\pi\tau)^3+(1-i2\pi\tau)^3)}{(1+4(\pi\tau)^2)^3}$. Since $(1 + i2\pi\tau)^3 = 1+3i2\pi\tau+3(i2\pi\tau)^2+(i2\pi\tau)^3$ and $(1-i2\pi\tau)^3 = 1-3i2\pi\tau+3(i2\pi\tau)^2-(i2\pi\tau)^3$ we get that $(1 + i2\pi\tau)^3 + (1 + i2\pi\tau)^3 = 2 + 3(i2\pi\tau)^2 = 2 - 12\pi^2\tau^2$ and thus the covariance function is $r(\tau) = \frac{4(1-6\pi^2\tau^2)}{(1+4\pi^2\tau^2)^3}$.

   (b) The spectral density of the filtered signal is $R_Y(f) = |H(f)|^2 R_X(f)$ where $H(f)$ is the transfer function is the transfer function $= \int_{\mathbb{R}} h(u)e^{-i2\pi f u}du = \int_{\mathbb{R}}(\delta_{-1}(u)+2\delta_0(u)+\delta_1(u))e^{-i2\pi f u}du = e^{-i2\pi f\cdot(-1)}+2e^0 e^{-i2\pi f\cdot 1} = \cos(2\pi f) + i\sin(2\pi f)+2+\cos(-2\pi f)+i\sin(-2\pi f) = 2(\cos(2\pi f)+1)$. Thus $R_Y(f) = |2(\cos(2\pi f) + 1)|^2 f^2 e^{-|f|} = 4f^2(\cos(2\pi f) + 1)^2 e^{-|f|}$. □

4. Calculate $C(X_1, X_2)$ if $\{X_t\}$ is shot noise with parameter 0.1 and pulse function

$$g(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

(3p)

   Solution: Using the indicator function $I(A)$, which is 1 if A is true and 0 otherwise, we have that $C(X_1, X_2) = r(1) = 0.1\int_{\mathbb{R}} g(u)g(u-1)\,du = 0.1\int_{\mathbb{R}} e^{-u} I(u > 0)\,e^{-(u-1)}I(u-1 > 0)\,du = 0.1\int_1^\infty e^{-u}e^{-u+1}du = 0.1e\int_1^\infty e^{-2u}du = 0.1e[-\frac{1}{2}\,e^{-2u}]_1^\infty = 0.05e^{-1} = 0.01839$. □

5. Assume that the random process $\{X_t : t \in \mathbb{R}\}$ has expectation function $m(t) = 1$ and covariance function $r(\tau) = e^{-\tau^2/2}$, $\tau \in \mathbb{R}$.

(a) Determine the spectral density function of $\{X_t\}$. (3p)

(b) Assuming that $\{X_t\}$ is a Gaussian process with $E(X_t) = 0$, calculate the probability $P(X_t + X_{t+1} \leq 1)$. (4p)

(c) The process is to be sampled at times $0, d, 2d, 3d, \ldots$ Determine the sample interval, d, such that the risk of aliasing is smaller than 5%. (4p)

Solution:

(a) According to the tables with $\alpha = \frac{1}{2}$ we get that $R(f) = \sqrt{2\pi}e^{-2\pi^2 f^2}$.

(b) $E(X_t+X_{t+1}) = 1+1 = 2$. $V(X_t) = r(0) = e^0 = 1$ and $C(X_t, X_{t+1}) = r(1) = = e^{-1/2}$. Therefore $V(X_t + X_{t+1}) = V(X_t) + V(X_{t+1}) + 2C(X_t, X_{t+1}) = = 2 + 2e^{-1/2}$ and $P(X_t + X_{t+1} \leq 1) = P\Big(\frac{X_t+X_{t+1}-2}{\sqrt{2+2e^{-1/2}}} \leq \frac{1-2}{\sqrt{2+2e^{-1/2}}}\Big) = = \Phi\Big(-\frac{1}{\sqrt{2+2e^{-1/2}}}\Big) = 1 - \Phi(0.56) = 0.2877.$

(c) From the theorem about the relation between the covariance functions and spectral densities of the signal and and sampled signal we can conclude that the risk of aliasing is $4\int_{1/2d}^{\infty} R(f)\, df$. To keep this below 5% we must have $\int_{1/2d}^{\infty} \sqrt{2\pi}e^{-(2\pi f)^2/2}df \leq 0.0125$ where the left side with the substitution $u = 2\pi f$ is $\sqrt{2\pi}\int_{\pi/d}^{\infty} e^{-u^2/2}(\frac{1}{2\pi}du) = \int_{\pi/d}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du$ and since $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the standard normal density function which integrates to 1 we have that $1 - \int_{-\infty}^{\pi/d} \phi(f)\, df \leq 0.0125$. Using the standard normal distribution function $\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt$ we have that $\Phi(\frac{\pi}{d}) \geq 0.9875$ and using the tables this means that $\frac{\pi}{d} \geq 2.28$, i.e. the sample interval d should be smaller than $\frac{\pi}{2.28} = 1.3779$. □