SOLUTIONS TO EXAM FOR STOCHASTIC MODELS IN DISCRETE TIME 3.75 ECTS

Master's program of Financial Mathematics
January 9, 2010, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Show that if $\mathbb{C}^*$ is the upper price of hedging against some non-negative $\mathcal{F}_N$-measurable function f_N and a contract is sold for a price greater than $\mathbb{C}^*$, then there exists an opportunity for arbitrage for the seller. (5p)

   Solution: (See p 396 in *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Let $\{N_t : t \in \mathbb{R}^+\}$ and $\{K_t : t \in \mathbb{R}^+\}$ be independent Poisson processes, both with intensity 1, and let the process $\{M_t : t = 1, 2, 3, \ldots\}$ be a process defined by $M_t = N_t - K_t$ for $t = 1, 2, 3, \ldots$.
   (a) Show that $\{M_t\}$ is a martingale with respect to the flow $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma(M_1, M_2, \ldots, M_t)$. (5p)
   (b) Calculate $E(M_t^2)$. (4p)

   Solution:
   (a) $E(|M_t|) = E(|N_t - K_t|) \leq E(|N_t|) + E(|K_t|) = 2$ (since both $N_t \geq 0$ and $K_t \geq 0$). Clearly all M_t are $\mathcal{F}_t$ measurable. Finally $E(M_{t+1}|\mathcal{F}_t) = E(N_{t+1} - K_{t+1}|\mathcal{F}_t) = E(N_{t+1} - N_t + N_t - (K_{t+1} - K_t + K_t)|\mathcal{F}_t) = E(N_{t+1} - N_t|\mathcal{F}_t) - E(K_{t+1} - K_t|\mathcal{F}_t) + E(N_t - K_t|\mathcal{F}_t) = E(N_{t+1} - N_t) - E(K_{t+1} - K_t) + E(M_t|\sigma(M_t, M_{t-1}, \ldots)) = 1 - 1 + M_t = M_t$.
   (b) $E(M_t^2) = E(N_t^2 - 2N_tK_t + K_t^2) \overset{N_t \perp K_t}{=} E(N_t^2) - 2E(N_t)E(K_t) + E(K_t^2)$. Now, $\lambda = 1$ so $E(N_t) = E(K_t) = t$ and $V(N_t) = V(K_t) = t$. Since $V(N_t) = E(N_t^2) - E(N_t)^2$ we have $E(N_t^2) = V(N_t) + E(N_t)^2 = t + t^2 = E(K_t^2)$ and thus $E(M_t^2) = t + t^2 - -2 \cdot t \cdot t + t + t^2 = 2t$. □

3. Calculate $C(X_t, X_{t+1})$ where $\{X_t : t \in \mathbb{Z}\}$ is an $ARMA(1,1)$ process. (4p)

   Solution: For an $ARMA(1,1)$ process we have that $X_{t+1} = aX_t + c\epsilon_t + \epsilon_{t+1}$. Thus $C(X_t, X_{t+1}) = C(X_t, aX_t + c\epsilon_t + \epsilon_{t+1}) = aV(X_t) + cC(X_t, \epsilon_t) + 0$. If we denote

$V(X_t) = \sigma_X^2$ and $V(\epsilon_t) = \sigma_\epsilon^2$ we get by $C(X_t, \epsilon_t) = C(aX_{t-1} + c\epsilon_{t-1} + \epsilon_t, \epsilon_t) = \sigma_\epsilon^2$ that $C(X_t, X_{t+1}) = a\sigma_X^2 + c\sigma_\epsilon^2$. □

4. $\{X_t : t \in \mathbb{Z}\}$ is a process according to the $ARCH(1)$ model. Assume a sample $x_1, x_2, \ldots, x_n$ is to be observed. For inference issues it is of interest to know its joint density function. Determine the joint density function conditional on the initial value, x_0, i.e. determine $f(x_1, x_2, \ldots, x_n | x_0)$. (6p)

   Solution: $X_t = \sigma_t \epsilon_t$ where $\sigma_t^2 = a_0 + a_1 X_{t-1}^2$ and the sequence $\{\epsilon_t\}$ is independent and $\epsilon_t \in N(0,1)$. Thus $f(x_1, x_2, \ldots, x_n | x_0) = \prod_{t=1}^{n} f(x_t | x_{t-1}, \ldots, x_0) = \prod_{t=1}^{n} f(x_t | x_{t-1})$. We have that $P(X_t \le x_t | X_{t-1} = x_{t-1}) =$ $= P(\sqrt{a_0 + a_1 X_{t-1}^2} \le x_t | X_{t-1} = x_{t-1}) = P\Big(\epsilon_t \le \frac{x_t}{\sqrt{a_0 + a_1 x_{t-1}^2}}\Big) = \Phi\Big(\frac{x_t}{\sqrt{a_0 + a_1 x_{t-1}^2}}\Big)$ so $f(x_t | x_{t-1}) = \frac{d}{dx_t} P(X_t \le x_t | X_{t-1} = x_{t-1}) = \frac{1}{\sqrt{a_0 + a_1 x_{t-1}^2}} \phi\Big(\frac{x_t}{\sqrt{a_0 + a_1 x_{t-1}^2}}\Big)$ where $\phi(x) = \frac{1}{2\pi} e^{-x^2/2}$. Therefore we have that $f(x_t | x_{t-1}) =$ $= \frac{1}{\sqrt{a_0 + a_1 x_{t-1}^2}} \cdot \frac{1}{2\pi} e^{-\left(\frac{x_t}{\sqrt{a_0 + a_1 x_{t-1}^2}}\right)^2/2} = \frac{1}{\sqrt{2\pi(a_0 + a_1 x_{t-1}^2)}} e^{-x_t^2/(2(a_0 + a_1 x_{t-1}^2))}$ and so the joint density function is $f(x_1, x_2, \ldots, x_n | x_0) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi(a_0 + a_1 x_{t-1}^2)}} e^{-x_t^2/(2(a_0 + a_1 x_{t-1}^2))} =$ $= (2\pi)^{-n/2} \prod_{t=1}^{n} (a_0 + a_1 x_{t-1}^2)^{-1/2} e^{-\frac{1}{2}\sum_{t=1}^{n} x_t^2/(a_0 + a_1 x_{t-1}^2)}$. □

5. An insurance company uses the risk process

$$R_t = u + 2t - \sum_{k=1}^{N_t} \zeta_k$$

   for their customers, where u is the initial capital, $\{N_t\}$ is a Poisson process with intensity 3, and $\{\zeta_k\}$ is a sequence of independent variables all exponentially distributed with intensity 2. Determine how large initial capital is needed for the risk of ruin to be less than 1%; i.e. how large u is needed for $P(\inf\{t : R_t \le 0\} < \infty) \le 0.01$? (6p)

   Solution: $R_t = 1 + 2t - \sum_{k=1}^{N_t} \zeta_k$ where $\{N_t\}$ is a Poisson process with intentsity $\lambda = 3$ and $\{\zeta_k\}$ are independent and Exponentially distributed with $\mu = 2$. According to the Lundberg-Cramér theorem, $P(\inf\{t : R_t \le 0\} < \infty) \le e^{-Ru}$ if there is an $R > 0$ such that $\int_0^\infty e^{Rx}(1 - F_\zeta(x))\,dx \le \frac{mu}{\lambda} = \frac{2}{3}$ where u is the initial capital. Since ζ_k are Exponentially distributed we have that $\int_0^\infty e^{Rx}(1 - F_\zeta(x))\,dx = \int_0^\infty e^{Rx}(1 - (1 - e^{-2x}))\,dx = \int_0^\infty e^{-(2-R)x} dx = [-\frac{1}{2-R} e^{-(2-R)x}]_0^\infty = \frac{1}{2-R} \le \frac{2}{3}$ if $R < 2$, which means that $3 \le 2(2-R) = 4 - 2R$, i.e. $R \le \frac{1}{2}$. Thus it is sufficient with $R = \frac{1}{2}$ for $P(\inf\{t : R_t \le 0\} < \infty) \le e^{-\frac{1}{2}u}$ which is ≤ 0.01 if $-\frac{1}{2}u \le \ln\frac{1}{100} = -\ln 100$, i.e. $u \ge 2\ln 100 \approx 9.210344$. □