Exam for Stochastic models in Discrete time
3.75 ECTS

Master's program of Financial Mathematics
January 9, 2010, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

For each problem a *complete* solution should be given. All solutions should be thoroughly presented. Each solution should start at the top of a new sheet of paper. Only one solution a sheet.
The proper solutions will be available on the internet at `http://www.hh.se/staff/erja` → `Teaching` → `Financial Mathematics` → `Stochastic models` → `Previous exams`

1. Show that if $\mathbb{C}^*$ is the upper price of hedging against some non-negative $\mathcal{F}_N$-measurable function f_N and a contract is sold for a price greater than $\mathbb{C}^*$, then there exists an opportunity for arbitrage for the seller. (5p)

2. Let $\{N_t : t \in \mathbb{R}^+\}$ and $\{K_t : t \in \mathbb{R}^+\}$ be independent Poisson processes, both with intensity 1, and let the process $\{M_t : t = 1, 2, 3, \ldots\}$ be a process defined by $M_t = N_t - K_t$ for $t = 1, 2, 3, \ldots$.
   (a) Show that $\{M_t\}$ is a martingale with respect to the flow $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \sigma(M_1, M_2, \ldots, M_t)$. (5p)
   (b) Calculate $E(M_t^2)$. (4p)

3. Calculate $C(X_t, X_{t+1})$ where $\{X_t : t \in \mathbb{Z}\}$ is an $ARMA(1, 1)$ process. (4p)

4. $\{X_t : t \in \mathbb{Z}\}$ is a process according to the $ARCH(1)$ model. Assume a sample $x_1, x_2, \ldots, x_n$ is to be observed. For inference issues it is of interest to know its joint density function. Determine the joint density function conditional on the initial value, x_0, i.e. determine $f(x_1, x_2, \ldots, x_n|x_0)$. (6p)

5. An insurance company uses the risk process

$$R_t = u + 2t - \sum_{k=1}^{N_t} \zeta_k$$

   for their customers, where u is the initial capital, $\{N_t\}$ is a Poisson process with intensity 3, and $\{\zeta_k\}$ is a sequence of independent variables all exponentially distributed with intensity 2. Determine how large initial capital is needed for the risk of ruin to be less than 1%; i.e. how large u is needed for $P(\inf\{t : R_t \leq 0\} < \infty) \leq 0.01$? (6p)

GOOD LUCK!