Solutions to Exam for Random Processes, 7.5 ECTS

January 16, 2010, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Prove that $G = \mathcal{F}(g)$, $h(\tau) \equiv G(\tau) \;\Rightarrow\; H(f) \equiv g(-f)$ (where $H = \mathcal{F}(h)$ and $g = \mathcal{F}^{-1}(G)$). (4p)

   Solution: (See the document linked from the course homepage for a proof of this result.) □

2. Provet that if $\{X_t : t \in \mathbb{R}\}$ is differentiable, then $E(X_t') = 0$. (3p)

   Solution: (See page 100 in the course literature.) □

3. Let $\{X_t : t = 1, 2, 3, \ldots\}$ be defined by $X_t = \frac{N_t - \lambda t}{\sqrt{\lambda t}}$, $t = 1, 2, 3, \ldots$ where $\{N_t : t \in \mathbb{R}\}$ is a Poisson process with intensity λ.
   (a) Calculate $P(X_2 \leq 1)$ if $\lambda = 3$. (3p)
   (b) Show that X_t is asymptotically standard normally distributed, i.e. give some arguments to why $\lim_{t\to\infty} P(X_t \leq x) = \Phi(x)$. (4p)

   Solution:
   (a) $P(X_2 \leq 1) = P(\frac{N_2 - 2\cdot 3}{\sqrt{2\cdot 3}} \leq 1) = P(N_2 \leq \sqrt{6}+6) = P(N_2 \leq 8.449\ldots) \overset{N_2 \in Poi(6)}{=} P(N_2 \leq 8) = 0.847$ (from the table).
   (b) $X_t = \frac{N_t - \lambda t}{\sqrt{\lambda t}} = \frac{\sum_{k=1}^t Z_k - \lambda t}{\sqrt{\lambda t}}$ where $\{Z_k\}$ is a sequence of independent variables, each distributed $Poi(\lambda)$. Thus $\{Z_k\}$ could be considered to be a sample of a variable $Z \in Poi(\lambda) \;\Rightarrow\; E(Z) = \lambda$ and $V(Z) = \lambda$. Therefore, according to the Central Limit Theorem (see the summary of formulae) we have that $P(\frac{\sqrt{t}}{\sigma}(\bar{Z} - \mu) \leq x) \to \Phi(x)$ as $t \to \infty$ where $\bar{Z} = \frac{1}{t}\sum_{k=1}^t Z_k$, $\mu = E(Z) = \lambda$, $\sigma = \sqrt{V(Z)} = \sqrt{\lambda}$ and $\Phi(x)$ is the standard normal distribution function. Rewriting this we get $P(\frac{\sqrt{t}}{\sigma}(\bar{Z} - \mu) \leq x) = P\left(\sqrt{\frac{t}{\lambda}}(\frac{1}{t}\sum_{k=1}^t Z_k - \lambda) \leq x\right) = P(\frac{1}{\sqrt{\lambda t}}(N_t - \lambda t) \leq x) = P(X_t \leq x)$. Thus X_t is asymptotically standard normally distributed. □

4. Let $\{X_t : t \in \mathbb{Z}\}$ be an $AR(1)$ processs with $a_1 = -\frac{2}{3}$ and $\epsilon_t \in N(0, \frac{1}{2})$ (i.e. $V(\epsilon_t) = \frac{1}{2}$). Calculate $E(X_t)$ and $V(X_t)$. (3p)

Solution: For the $AR(1)$ process with $a_1 = -\frac{2}{3}$ we have $X_t - \frac{2}{3}X_{t-1} = \epsilon_t \Rightarrow X_t = \frac{2}{3}X_{t-1} + \epsilon_t$. Therefore $m = E(X_t) = E(\frac{2}{3}X_{t-1} + \epsilon_t) = \frac{2}{3}E(X_t) + E(\epsilon_t) = \frac{2}{3}m + 0 \Rightarrow E(X_t) = 0$ and $\sigma^2 = V(X_t) = C(\frac{2}{3}X_{t-1} + \epsilon_t, \frac{2}{3}X_{t-1} + \epsilon_t) = (\frac{2}{3})^2 V(X_{t-1}) + 2 \cdot \frac{2}{3}C(X_{t-1}, \epsilon_t) + V(\epsilon_t) = \frac{4}{9}\sigma^2 + 0 + \frac{1}{2} \Rightarrow \sigma^2 = \frac{1/2}{1-4/9} = 0.9$. □

5. The process $\{Y_t : t \in \mathbb{R}\}$ is weakly stationary with spectral density $R_Y(f) = |f|e^{-|f|}$. Now, when sampling Y_t one wants to choose the sampling distance such that the boundary value of the spectral density of the sampled process, $\{Z_t\}$, is not greater than 1, i.e. if one samples at $0, \pm d, \pm 2d, \pm 3d, \ldots$ the number d is to be chosen such that $R_Z(\frac{1}{2d}) \leq 1$. What value should d have? (4p)

Solution: Since $R_Z(f)$ is non-zero only for $-\frac{1}{2d} < f \leq \frac{1}{2d}$, we have that

$$
\begin{aligned}
R_Z(f) &= \sum_{k=-\infty}^{\infty} R_Y(f + \tfrac{k}{d}) \\
&= \sum_{k=-\infty}^{\infty} |f + \tfrac{k}{d}|e^{-|f+k/d|} \\
&= \sum_{k=-\infty}^{-1} -(f + \tfrac{k}{d})e^{f+k/d} + |f + \tfrac{0}{d}|e^{-|f+0/d|} + \sum_{k=1}^{\infty}(f + \tfrac{k}{d})e^{-(f+k/d)} \\
&= e^f\Big(- f\sum_{k=1}^{\infty} e^{-k/d} + \sum_{k=1}^{\infty} \tfrac{k}{d}e^{-k/d}\Big) + |f|e^{-|f|} + e^{-f}\Big(f\sum_{k=1}^{\infty} e^{-k/d} + \sum_{k=1}^{\infty} \tfrac{k}{d}e^{-k/d}\Big) \\
&= \tfrac{1}{d}(e^f + e^{-f})\sum_{k=1}^{\infty} k(e^{-1/d})^k + |f|e^{-|f|} - f(e^f - e^{-f})\sum_{k=1}^{\infty}(e^{-1/d})^k \\
&= \frac{1}{e^{1/d} - 1}\left(\frac{2\cosh f}{d(1 - e^{-1/d})} - 2\sinh f\right) + |f|e^{-|f|}
\end{aligned}
$$

where $2\cosh f = e^f + e^{-f}$, $2\sinh f = e^f - e^{-f}$ and the last step of the calculation is due to the geometric series $\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$ and its derivative $\sum_{k=1}^{\infty} ka^k = a\sum_{k=1}^{\infty} ka^{k-1} = a\frac{d}{da}\sum_{k=1}^{\infty} a^k = a\frac{d}{da}(\frac{a}{1-a}) = \frac{a}{(1-a)^2}$.

By calculating $R_Z(\frac{1}{2d})$, for some values of d, one finds that $d \leq 0.43992$ will give a boundary value of the spectral density below 1. □

6. Let $\{\xi_t : t \in \mathbb{R}\}$ be weakly stationary process with cvf $r(\tau) = \frac{\cos\tau}{1+\tau^2}$.

(a) Prove that $\{\xi_t\}$ is differentiable in squared mean. (3p)

(b) Calculate $P(\xi_t^2 \leq 2)$ assuming that $\{\xi_t\}$ is Gaussian with $m_\xi = 1$. (3p)

(c) Assume the input $\{\xi_t\}$ is filtered using a filter with impulse response $h(t) = \frac{1}{\pi}\delta_{-\pi}(t) + \frac{1}{\pi}\delta_\pi(t)$. Determine the cvf of the output signal. (4p)

Solution:

(a) $r(\tau) = \frac{\cos\tau}{1+\tau^2} \Rightarrow r'(\tau) = \frac{(\cos\tau)\cdot 2\tau - (-\sin\tau)(1+\tau^2)}{(1+\tau^2)^2} = \frac{2\tau\cos\tau + (1+\tau^2)\sin\tau}{(1+\tau^2)^2} \Rightarrow$ $r''(\tau) = \ldots = \frac{8\tau^2\cos\tau}{(1+\tau^2)^3} + \frac{4\tau\sin\tau}{(1+\tau^2)^2} + \frac{(3+\tau^2)\cos\tau}{1+\tau^2}$ which is continuous for all $\tau \in \mathbb{R}$, i.e. r is twice differentiable, i.e. $\{\xi_t\}$ is differentiable in squared mean.

(b) $P(\xi_t^2 \leq 2) = P(-\sqrt{2} \leq \xi_t \leq \sqrt{2}) = \underbrace{P(\xi_t \leq \sqrt{2})}_{I} - \underbrace{P(\xi_t \leq -\sqrt{2})}_{II}$.

Here $r(\tau) = \frac{\cos\tau}{1+\tau^2} \Rightarrow V(\xi_t) = r(0) = 1 \Rightarrow I = P(\xi_t \leq \sqrt{2}) = P(\frac{\xi_t - 1}{1} \leq \frac{\sqrt{2}-1}{1}) = \Phi(\sqrt{2}-1) = 0.6591$ and $II = P(\xi_t \leq -\sqrt{2}) = \Phi(-\sqrt{2}-1) = 1 - \Phi(\sqrt{2}+1) = 0.008$. Thus $P(\xi_t^2 \leq 2) = 0.6591 - 0.008 = 0.6511$.

(c) $r_Y(\tau) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v) r_\xi(\tau + u - v)\, du\, dv =$

$= \int_{-\infty}^{\infty} (\frac{1}{\pi}\delta_{-\pi}(u) + \frac{1}{\pi}\delta_\pi(u)) \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \delta_{-\pi}(v) \frac{\cos(\tau+u-v)}{1+(\tau+u-v)^2}\, dv + \frac{1}{\pi}\int_{-\infty}^{\infty} \delta_\pi(v) \frac{\cos(\tau+u-v)}{1+(\tau+u-v)^2}\, dv\right) du =$

$\frac{1}{\pi}\left(\frac{1}{\pi} \cdot \frac{\cos(\tau-\pi+\pi)}{1+(\tau-\pi+\pi)^2} + \frac{1}{\pi} \cdot \frac{\cos(\tau-\pi-\pi)}{1+(\tau-\pi-\pi)^2}\right) + \frac{1}{\pi}\left(\frac{1}{\pi} \cdot \frac{\cos(\tau+\pi+\pi)}{1+(\tau+\pi+\pi)^2} + \frac{1}{\pi} \cdot \frac{\cos(\tau+\pi-\pi)}{1+(\tau+\pi-\pi)^2}\right) =$

$\frac{\cos\tau}{\pi^2}\left(\frac{2}{1+\tau^2} + \frac{1}{1+(\tau+2\pi)^2} + \frac{1}{1+(\tau-2\pi)^2}\right)$. □