Solutions to Exam for Random Processes, 7.5 ECTS

December 20, 2008, 9.00 am – 1.00 pm

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and Math. Handbook: Beta.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Assume $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is distributed according to the multivariate normal distribution. Show that if $C(X_i, X_j) = 0$ for all $i \neq j$, then X_i is independent of X_j for all $i \neq j$. (4p)

   Solution: If $X_1, X_2, \ldots, X_n$ are uncorrelated, then the covariance matrix, Σ, consists of elements

$$\sigma_{ij} = \begin{cases} \sigma_{ii} = V(X_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

   and $\det(\Sigma) = \prod_{i=1}^n \sigma_{ii}$ and the inverse of Σ is Σ^{-1} consisting of elements

$$\theta_{ij} = \begin{cases} 1/\sigma_{ii} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

   Thus the joint density of $\boldsymbol{x}$ is

$$\begin{aligned} f(\boldsymbol{x}) &= f(x_1, x_2, \ldots, x_n) \\ &= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\Big(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\Big) \\ &= \frac{1}{\prod_{i=1}^n \sqrt{2\pi\sigma_{ii}}} \exp\Big(-\tfrac{1}{2}\textstyle\sum_{i=1}^n (x_i-\mu_i)\frac{1}{\sigma_{ii}}(x_i-\mu_i)\Big) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_{ii}}} \exp\Big(-\tfrac{1}{2}(x_i-\mu_i)^2/\sigma_{ii}\Big) \\ &= \prod_{i=1}^n f(x_i) \end{aligned}$$

   where $f(x_i)$ is the density of $X_i \in N(\mu_i, \sigma_{ii})$. Thus $\{X_i : i = 1, 2, \ldots, n\}$ are independent. □

2. Suppose $\{X_t : t \in \mathbb{R}\}$ is a Poisson process with parameter $\lambda = 2$. Calculate
   (a) $E(X_t + 2X_{t+1})$, (3p)
   (b) the covariance function of $\{X_t\}$. (4p)

Solution:

(a) $E(X_t + 2X_{t+1}) = E(X_t) + 2E(X_{t+1}) = 2t + 2 \cdot 2(t+1) = 6t + 4$.

(b) Since the increments are independent we have that $C(X_s, X_t) \overset{s\leq t}{=} C(X_s, X_t - X_s + X_s) = C(X_s, X_t - X_s) + C(X_s, X_s) = 0 + V(X_s) = 2s$. Similarly $C(X_s, X_t) \overset{s\geq t}{=} C(X_s - X_t + X_t, X_t) = C(X_s - X_t, X_t) + C(X_t.X_t) = 0 + 2t$. Thus $C(X_s, X_t) = \left\{ \begin{array}{ll} 2s & \text{if } s < t \\ 2t & \text{if } t \leq s \end{array} \right\} = 2\min(s,t)$. □

3. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process with expectation function $m(t) = 1$ and covariance function $r(\tau) = e^{-2|\tau|}$. Determine
   (a) $V(X_t)$, (2p)
   (b) $P(X_{t+1} - X_t > 1)$, (4p)
   (c) the spectral density function of $\{X_t\}$. (3p)

Solution:

(a) $V(X_t) = r(0) = e^0 = 1$.

(b) Since $\{X_t\}$ is a Gaussian process the linear combination $X_{t+1} - X_t$ is normally distributed with parameters μ and σ^2 where $\mu = E(X_{t+1} - X_t) = E(X_{t+1}) - E(X_t) = 1 - 1 = 0$ and $\sigma^2 = V(X_{t+1} - X_t) = C(X_{t+1} - X_t, X_{t+1} - X_t) = C(X_{t+1}, X_{t+1}) - C(X_{t+1}, X_t) - C(X_t, X_{t+1}) + C(X_t, X_t) = V(X_{t+1}) - 2C(X_{t+1}, X_t) + V(X_t) = 2r(0) - 2r(1) = 2(1 - e^{-2}) \Rightarrow P(X_{t+1} - X_t > 1) = 1 - P(X_{t+1} - X_t \leq 1) = 1 - P(\frac{X_{t+1} - X_t - \mu}{\sigma} \leq \frac{1-\mu}{\sigma}) = 1 - \Phi(\frac{1-0}{\sqrt{2(1-e^{-2})}}) = 1 - \Phi(0.7604) = 0.2236$.

(c) The spectral density is the Fourier transform of the covariance function: $R(f) = \mathcal{F}(r)(f)$. According to the tables $\mathcal{F}(e^{-\alpha|\tau|}) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$. Thus with $\alpha = 2$ we get $R(f) = \frac{2 \cdot 2}{2^2 + 4\pi^2 f^2} = \frac{1}{1 + \pi^2 f^2}$. (If one does not want to use the tables one can make the calculation but this is of course a detour[1]). □

[1] Then the calculation would have been: $R(f) = \int_{\mathbb{R}} e^{-i2\pi f\tau} e^{-2|\tau|} d\tau = \int_{-\infty}^{0} e^{-i2\pi f\tau + 2\tau} d\tau + \int_0^\infty e^{-i2\pi f\tau - 2\tau} d\tau \overset{u=-\tau}{=} -\int_\infty^0 e^{-(i2\pi f + 2)u} du + \int_0^\infty e^{-(i2\pi f+2)\tau} \int_0^\infty e^{-2u} 2\cos(2\pi f u)\, du \overset{P.I.}{=} 2(-0 + \frac{1}{2} - \pi f \int_0^\infty (-\frac{e^{-2u}}{2})(-2\pi f \sin(2\pi f u))\, du) \overset{P.I.}{=} 1 - 2\pi f(-0 + 0 + \frac{\pi f}{2} \int_0^\infty e^{-2u} 2\cos(2\pi f u)\, du)$. Now by giving the integral $\int_0^\infty e^{-2u} 2\cos(2\pi f u)\, du$ the name I we can write $I = 1 - 2\pi f \cdot \frac{\pi f}{2} I \Rightarrow I(1 + (\pi f)^2) = 1 \Rightarrow I = \frac{1}{1 + (\pi f)^2}$.

4. Suppose $\{X_t : t \in \mathbb{Z}\}$ is an $AR(1)$ process. Determine the value of the parameter σ_ϵ^2 if $V(X_t) = \sigma_X^2 = 1$ and $a_1 = -1/4$. (5p)

Solution: $\{X_t\}$ being an $AR(1)$ process we have that

$$X_t = -\tfrac{1}{4}X_{t-1} + \sigma_\epsilon \epsilon_t \qquad (1)$$

where $\{\epsilon_t\}$ are independent and $\epsilon_t \in N(0,1)$. Thus $1 = \sigma_X^2 = V(X_t) \overset{(1)}{=} V(-\frac{1}{4}X_{t-1} + \sigma_\epsilon\epsilon_t) = V(-\frac{1}{4}X_{t-1}) + V(\sigma_\epsilon\epsilon_t) + 2C(-\frac{1}{4}X_{t-1}, \sigma_\epsilon\epsilon_t) = \frac{1}{4^2}V(X_{t-1}) + \sigma_\epsilon^2 V(\epsilon_t) + 2\cdot 0 = \frac{1}{16}\sigma_X^2 + \sigma_\epsilon^2 \;\Rightarrow\; \sigma_\epsilon^2 = \frac{15}{16}$. □

5. Assume that the process $\{X_t : t \in \mathbb{R}\}$ is filtered with the impulse response $\delta'(t)$. What is the output signal? (5p)
Hint: For the delta function $\int_{\mathbb{R}} \delta'(u)f(u)\,du = \int_{\mathbb{R}} \delta(u)f'(u)\,du$. *Use this!*

Solution: For the delta function we have that $\int f'(x)\delta(x)\,dx = \int f(x)\delta'(x)\,dx$, and for any impulse response $h(u)$ of a linear time invariant filter H it holds that $\int_{\mathbb{R}} h(t-u)X_u\,du = \int_{\mathbb{R}} h(u)X_{t-u}\,du$. Thus the output signal from filtering with impulse response $h(u) = \delta'(u)$ is

$$\begin{aligned}
\int_{\mathbb{R}} h(u)X_{t-u}\,du &= \int_{\mathbb{R}} \delta'(u)X_{t-u}\,du \\
&= \int_{\mathbb{R}} \delta(u)X'_{t-u}\,du \\
&= \int_{\mathbb{R}} \delta(t-u)X'_u\,du \\
&= \int_{\mathbb{R}} \delta_t(u)X'_u\,du \\
&= X'_t
\end{aligned}$$

i.e. the derivative of the input signal! □