Stochastic models
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Exercises after Chapter II

1. Let $X_n = \sum_{k=1}^{n} I(B_k)$ where $B_k \in \mathcal{F}_k$ for all $k = 1, 2, 3, \ldots$ and $\{\mathcal{F}_n\}$ is a flow. Show that $\{X_n\}$ is a submartingale with respect to $\{\mathcal{F}_n\}$.

2. *Doob decomposition* Let $\{X_n : n \in \mathbb{Z}^+\}$ be an adapted integrable process and define the process $\{A_n\}$ by
$$A_n = \sum_{k=1}^{n} E(X_k - X_{k-1} | \mathcal{F}_{k-1})$$
Show that
   (a) $\{A_n\}$ is previsible.
   (b) $\{M_n\}$ is a martingale if $M_0 = X_0$ and $M_n = X_n - A_n$ for all $n = 1, 2, 3, \ldots$
   (c) $\{X_n\}$ is a submartingale iff $\{A_n\}$ is an increasing[1] previsible process[2].

3. [J.M. Steele, 2001] Assume $\{M_n\}$ is a martingale with respect to the flow $\{\mathcal{F}_n\}$ and that $E(M_n^2) < \infty$ for all n. Show that we may decompose M_n^2 into $N_n + A_n$ where $\{N_n\}$ is a martingale with respect to the flow $\{\mathcal{F}_n\}$, $\{A_n\}$ is previsible and non-decreasing (i.e. $A_n \in \{\mathcal{F}_{n-1}\}$ and $A_{n+1} \geq A_n$ for all n).

   Hint: Let $A_0 = 0$ and $A_{n+1} = A_n + E((M_{n+1} - M_n)^2 | \mathcal{F}_n)$ for all $n \geq 0$.

4. Let $\{N_t : t \in \mathbb{R}^+\}$ be a Possion process.
   (a) Show that $\inf\{t : N_t = 1\}$ is exponentially distributed.
   (b) Let $\{Y_k\}$ be observations of $\{N_t\}$ at integer times $t = k$, i.e. let $Y_k = N_k$, $k \in \mathbb{Z}^+$. Show that $\{Y_k\}$ is a submartingale with respect to the flow $\{\mathcal{F}_n\} = \{\sigma(Y_1, \ldots, Y_n)\}$.

5. Suppose that you play the roulette and repeatedly bet a small amount, c, on number 13 with the gain $35c$ if the wheel stops at number 13 and with the loss c if the wheel stops at some other number. Let M_n be the total amount gained by round n, $\{\mathcal{F}_n\} = \{\sigma(M_1, \ldots, M_n)\}$ and
   (a) show that $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ under the assumption that there are 36 numbers: $1, 2, \ldots, 36$ in the roulette wheel.
   (b) (more realistically) show that $\{M_n\}$ is a supermartingale with respect to $\{\mathcal{F}_n\}$ under the assumption that there are 37 numbers: $0, 1, 2, \ldots, 36$ in the roulette wheel.
   (c) calculate $E(\max(M_1, M_2))$ if there are 37 numbers in the roulette wheel.

[1] A random process is increasing if $A_n \leq A_{n+1}$ a.s. for all n.

[2] This is the famous Doob decomposition: *every submartingale is the sum of a martingale and an increasing previsible process.*

6. Let $\zeta_1, \zeta_2, \ldots$ be independent with $E(\zeta_i) = 0$ and $D(\zeta_i) = \sigma_i^2$. Further let $S_n^2 = \sum_{i=1}^n \zeta_i^2$ and $\varsigma_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $\{S_n^2 - \varsigma_n^2\}$ is a martingale with respect to the flow $\{\sigma(S_1^2, \ldots, S_n^2)\}$.

Definition A function $g : \mathbb{R}^n \to \mathbb{R}$ is **convex** if
$\forall \boldsymbol{x}_0 \in \mathbb{R}^n \ \exists \boldsymbol{a} \in \mathbb{R}^n, b \in \mathbb{R} : \boldsymbol{a}\boldsymbol{x}_0 + b = g(\boldsymbol{x}_0)$ and $\boldsymbol{a}\boldsymbol{x} + b \leq g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

7. (a) Show *Jensens inequality*, i.e. $g : \mathbb{R}^n \to \mathbb{R}$ convex $\Rightarrow E(g(\boldsymbol{x})) \geq g(E(\boldsymbol{x}))$.
   (b) Show that if $\{X_n\}$ and $\{Y_n\}$ are submartingales with respect to $\{\mathcal{F}_n^X\}$ and $\{\mathcal{F}_n^Y\}$ respectively, then $\{\max(X_n, Y_n)\}$ is also with respect to $\{\mathcal{F}_n\} = \{\sigma(\mathcal{F}_n^X, \mathcal{F}_n^Y)\}$.

Definition For a weakly stationary process, $\{X_n\}$, let R denote the **covariance function** $R(k) = Cov(X_n, X_{n-k})$.

8. Determine the covariance function R of an $MA(q)$ process.

9. Assume $\{X_n\}$ is an $AR(1)$ process with coefficent a and noise $\epsilon_t \in N(0, \sigma_\epsilon^2)$.
   (a) Determine the set of feasible values of a (so $\{X_n\}$ is weakly stationary).
   (b) Show that the covariance function is $R(k) = \frac{a^{|k|}\sigma_\epsilon^2}{1-a}$ by using induction.

10. Assume $\{X_n\}$ is an $AR(2)$ process with coefficients $a_1 = a_2 = a$.
    (a) Determine the set of feasible values of a.
    (b) What kind of process is $\{X_n - X_{n-1}\}$.
    (c) Relaxing the assumption that $a_1 = a_2$, what is the set of feasible values of a_1 and a_2.

11. In the case of an $AR(p)$ process with noise $\epsilon_t \in N(0, \sigma_\epsilon^2)$
    (a) derive the Yule-Walker equations:
    $$R(k) - a_1 R(k-1) - \ldots - a_p R(k-p) = \begin{cases} \sigma_\epsilon^2 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, 3, \ldots \end{cases}$$
    (b) show that it is a Markov chain. What are the transition probabilities?

12. In `http://dixon.hh.se/erja/teach/finmath/course01/data_for_exerc2.dat` are the log ask prices for the option trade with the Microsoft share in the NASDAQ market, for 660 consecutive time-points.
    (a) Calculate the Maximum Likelihood estimates (MLE's) of the parameters in an $ARIMA(1, 1, 0)$.
    (b) Assume the $ARIMA(1, 1, 0)$ model for these data and ML estimate its parameters. (The solution should be presented together with program code.)

13. In the $ARMA(1,1)$ model calculate the mean and variance under stationarity.

14. In the $AR(2)$ model derive the MLE of the parameters $\alpha_0, \alpha_1, \alpha_2$. The selling price of the share *Ericsson B* at *Stockholms fondbörs* during 2006-09-25 had the following record:

Hour	*Minute*	*Price*
8	00	25.2
	15	25.3
	30	25.3
	45	25.4
9	00	25.5
	15	25.35
	30	25.45
	45	25.35
10	00	25.25
	15	25.35
	30	25.35
	45	25.3
11	00	25.35
	15	25.25
	30	25.25
	45	25.3

Hour	*Minute*	*Price*
12	00	25.2
	15	25.25
	30	25.2
	45	25.25
13	00	25.25
	15	25.2
	30	25.2
	45	25.25
14	00	25.25
	15	25.2
	30	25.25
	45	25.15
15	00	25.2
	15	25.25
	30	25.2
	45	25.15

Source: Dagens industri 2006-09-25, `http://www.di.se`

Assuming an $AR(2)$ model for these data and estimate the expected value by first estimating the AR parameters. (The solution should be presented together with program code.)

15. In the $ARCH(1)$ model, assuming stationarity,
    (a) calculate the fourth moment,
    (b) which are the feasible values of the coefficients?
    (c) show that the distribution is *leptokurtic*, Assume that the observation $\{x_0, x_1, \ldots, x_n\}$ of an $ARCH(1)$ model is made.
    (d) Show that, conditional on $X_0 = x_0$, the Maximum likelikhood estimators of the parameters a_0 and a_1 are the values of these parameters which satisfy the equations

$$\begin{cases} \sum_{k=1}^{n} \frac{1}{a_0+a_1x_{k-1}^2} &= \frac{1}{2}\sum_{k=1}^{n} \frac{x_k^2}{(a_0+a_1x_{k-1}^2)^2} \\ \sum_{k=1}^{n} \frac{x_{k-1}^2}{a_0+a_1x_{k-1}^2} &= \frac{1}{2}\sum_{k=1}^{n} \frac{x_k^2x_{k-1}^2}{(a_0+a_1x_{k-1}^2)^2} \end{cases}$$

16. In the $GARCH(1,1)$ model, calculate, assuming stationarity,
   (a) the second and fourth moments.
   (b) which are the feasible values of the coefficients?

17. In the $HARCH(2)$ model, assuming stationarity,
   (a) calculate the first, second and fourth moments.
   (b) which are the feasible values of the coefficients?

18. Suppose $\{h_n\}$ is distributed according to the stochastic volatility model of first order (i.e. with $p = 1$). Show that
   (a) $\{h_n\}$ is a *martingale difference* with respect to the flow $\mathcal{F}_n^{\epsilon,\delta} = \sigma(\epsilon_1, \ldots, \epsilon_n, \delta_1, \ldots, \delta_n)$ (i.e. show that $E(h_{n+1}|\mathcal{F}_n^{\epsilon,\delta}) = 0$).
   (b) the process variables $\{h_n\}$ are uncorrelated.

19. Assume that $\{h_n\}$ is distributed according to the $HARCH(p)$ model with coefficients $a_0, a_1, \ldots, a_p$ all positive. Show that

$$\sum_{j=1}^{p} j\, a_j \;<\; 1$$