23. Let $\{R_t\}$ be a risk process $R_t = u + ct - \sum_{k=1}^{N_t} \xi_k$ where the term $u+ct$ stands for the (deterministic) baseline and premiums paid by insurance takers and claims $\{\xi_t\}$ occur at random time-points according to the Poisson process $\{N_t\}$. Make the simplifying assumptions that $u = 0$, $c = 1$, $\xi_k \equiv 1$ for all k and let the value of the intensity parameter of the Poisson process be 1. Assume that the process $\{R_t\}$ is observed at times $t = 1, 2, 3, \ldots$
   (a) Calculate its expectation function,
   (b) Calculate its covariance function,
   (c) Is the process strongly stationary?

   Let τ be the time of ruin, i.e. $\tau = \inf\{t \geq 1 : R_t \leq 0\}$.
   (d) Calculate the probability of ruin at time 3, i.e. $P(\tau = 3)$.
   (e) Calculate the general ruin probability function, i.e. $P(\tau = t)$.

24. [D. Williams, 1990] The *Riemann zeta function* ζ is defined by

$$\zeta(s) = \sum_{n \in \mathbb{Z}+} n^{-s}$$

   for $s > 1$. Let X and Y be independent positive integer valued random variables with $P(X = n) = P(Y = n) = n^{-s}/\zeta(s)$.
   (a) Let E_p be the event $\{X \text{ is divisible by } p\}$ and prove that the events $\{E_p : p \in \{\text{primes}\}\}$ are independent.
   (b) Explain Euler's formula

$$\frac{1}{\zeta(s)} = \prod_{p \in \{\text{primes}\}} (1 - p^{-s})$$

   from a probabilistic point of view.
   (c) Show that

$$P(\text{no square other that 1 divides } X) = \frac{1}{\zeta(2s)}.$$

   (d) Prove that if H is the greatest common divisor of X and Y, then

$$P(H = n) = \frac{1}{n^{2s}\zeta(2s)}.$$

23 a) $\xi_k \equiv 1 \Rightarrow \sum_{k=1}^{N_t} \xi_k = \sum_{k=1}^{N_t} 1 = N_t \Rightarrow$

$$\Rightarrow E(R_t) = E\left(u + ct - \sum_{k=1}^{N_t} \xi_k\right) =$$

$$= \underbrace{u}_{=0} + \underbrace{c}_{1}t - E(N_t) = t - \underbrace{\lambda}_{=1} t = 0$$

b) $Cov(R_s, R_t) = Cov(s - N_s, t - N_t) =$

$$= Cov(N_s, N_t) = \lambda \min(s,t) = \min(s,t)$$

c) Since $Cov(R_s, R_t) = \min(s,t)$ is <u>not</u> a function of $s-t$, the process $\{R_t\}$ is not weakly stationary. Since it is not weakly stationary, it is neither strongly stationary.

d) Since the process is observed in discrete time we represent the Poisson process as $N_t = \sum_{k=1}^{t} Z_k$ where $Z_k \in Poi(1)$ and independent

$P(\tau = 3) = P(R_1 > 0, R_2 > 0, R_3 \le 0) = P(1 - N_1 > 0,$
$2 - N_2 > 0, 3 - N_3 \ge 0) = P(Z_1 < 1, Z_1 + Z_2 < 2, Z_1 + Z_2 + Z_3 \ge 3)$

$$= P(Z_1 = 0)\Big(P(Z_2 = 0, Z_3 \ge 3) + P(Z_2 = 1, Z_3 \ge 3 - 1)\Big)$$

$$= P(Z_1 = 0)\Big(P(Z_2 = 0)(1 - P(Z_3 \le 2)) + P(Z_2 = 1)(1 - P(Z_3 \le 1))\Big)$$

$$= \frac{1^0}{0!}e^{-1}\left(e^{-1}\left(1 - \frac{1^0}{0!}e^{-1} - \frac{1^1}{1!}e^{-1} - \frac{1^2}{2!}e^{-1}\right) + \frac{1^1}{1!}e^{-1}\left(1 - \frac{1^0}{0!}e^{-1} - \frac{1^1}{1!}e^{-1}\right)\right)$$

$$= e^{-3}\big((e - 2.5) + (e - 2)\big) = 0.04663$$

e) $P(\tau = t) = P(R_1 > 0, \dots, R_{t-1} > 0, R_t \leq 0)$

$= P(1 - N_1 > 0, 2 - N_2 > 0, \dots, t-1-N_{t-1} > 0, t - N_t \leq 0)$

$= P(Z_1 = 0, Z_1 + Z_2 \leq 1, \dots, Z_1 + \dots + Z_{t-1} \leq t-2, Z_1 + \dots + Z_t \geq t)$

$= P(Z_1 = 0)\, P(Z_2 \leq 1, \dots, Z_2 + \dots + Z_{t-1} \leq t-2, Z_2 + \dots + Z_t \geq t)$

$= e^{-1} \sum_{z_2=0}^{1} P(Z_2 = z_2, \dots, Z_3 + \dots + Z_{t-1} \leq t-2-z_2, Z_3 + \dots + Z_t \geq t - z_2)$

$= e^{-1} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} P(Z_4 + \dots + Z_{t-1} \leq t-2-z_2-z_3, \dots, Z_4 + \dots + Z_t \geq t - z_2 - z_3)$
$P(Z_2 = z_2)\, P(Z_3 = z_3)$

$= e^{-1} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} \cdots \sum_{z_{t-1}=0}^{t-1-z_2-\dots-z_{t-2}} P(Z_t \geq t - z_2 - \dots - z_{t-1}) \prod_{k=2}^{t-1} P(Z_k = z_k)$

$= e^{-1} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} \cdots \sum_{z_{t-1}=0}^{t-1-z_2-\dots-z_{t-2}} (1 - P(Z_t < t - z_2 - \dots - z_{t-1})) \prod_{k=2}^{t-1} P(Z_k = z_k)$

$= e^{-1} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} \cdots \sum_{z_{t-1}=0}^{t-1-z_2-\dots-z_{t-2}} \left(1 - \sum_{z_t=0}^{t-1-z_2-\dots-z_{t-1}} P(Z_t = z_t)\right) \prod_{k=2}^{t-1} P(Z_k = z_k)$

$= e^{-1} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} \cdots \sum_{z_{t-1}=0}^{t-1-z_2-\dots-z_{t-2}} \left(1 - \sum_{z_t=0}^{t-1-z_2-\dots-z_{t-1}} \frac{1}{z_t!} e^{-1}\right) \prod_{k=2}^{t-1} \frac{1}{z_k!} e^{-1}$

$= e^{-t} \sum_{z_2=0}^{1} \sum_{z_3=0}^{2-z_2} \cdots \sum_{z_{t-1}=0}^{t-1-z_2-\dots-z_{t-2}} \left(e - \sum_{z_t=0}^{t-1-z_2-\dots-z_{t-1}} \frac{1}{z_t!}\right) \prod_{k=2}^{t-1} \frac{1}{z_k!}$

24 $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ where $s > 1$

X and Y independent with
$P(X=n) = P(Y=n) = n^{-s}/\zeta(s)$ for all $n \in \mathbb{Z}^+$.

a) Let E_p be the event $\{p \text{ divides } X\}$. Now the events $E_2, E_3, E_5, \dots$ are independent iff for all prime number sequences $p_1, p_2, \dots, p_n$ we have that

$$P(E_{p_1} \cap E_{p_2} \cap \dots \cap E_{p_n}) = P(E_{p_1}) P(E_{p_2}) \cdots P(E_{p_n})$$

$$P(E_{p_1}) = P(p_1 \text{ divides } X)$$
$$= P(\{X = p_1\} \cup \{X = 2p_1\} \cup \{X = 3p_1\} \cup \dots)$$
$$= \sum_{k=1}^{\infty} P(X = kp_1) = \sum_{k=1}^{\infty} \frac{(kp_1)^{-s}}{\zeta(s)} =$$
$$= \frac{p_1^{-s}}{\zeta(s)} \underbrace{\sum_{k=1}^{\infty} k^{-s}}_{\zeta(s)} = p_1^{-s}$$

$$\Rightarrow P(E_{p_1} \cap \dots \cap E_{p_n}) = P(\{p_1 \text{ divides } X\} \cap \dots \cap \{p_n \text{ divides } X\}) =$$
$$= P(\{X = p_1 \cdots p_n\} \cup \{X = 2p_1 \cdots p_n\} \cup \{X = 3p_1 \cdots p_n\} \cup \dots)$$
$$= \sum_{k=1}^{\infty} P(X = kp_1 \cdots p_n) = \sum_{k=1}^{\infty} \frac{(kp_1 \cdots p_n)^{-s}}{\zeta(s)} = \frac{(p_1 \cdots p_n)^{-s}}{\zeta(s)} \sum_{k=1}^{\infty} k^{-s} =$$
$$= p_1^{-s} p_2^{-s} \cdots p_n^{-s} = P(E_{p_1}) P(E_{p_2}) \cdots P(E_{p_n}).$$

b) $\frac{1}{\zeta(s)} = P(X=1) = P(\text{no prime divides } X) =$

$$= P\Big(\bigcap_{p\in\{\text{primes}\}} E_p^c\Big) = \prod_{p\in\{\text{primes}\}} P(E_p^c) = \prod_{p\in\{\text{primes}\}} (1-P(E_p)) =$$

$$= \prod_{p\in\{\text{primes}\}} (1-p^{-s})$$

c) Show $P(\text{no square other than 1 divides } X) = \frac{1}{\zeta(2s)}$

Need only consider prime squares so

$P(\text{no prime square divides } X) = P(E_{2^2}^c \cap E_{3^2} \cap E_{5^2}^c \cap \cdots) =$

$$= P\Big(\bigcap_{p\in\{\text{primes}\}} E_{p^2}^c\Big) = \prod_{p\in\{\text{primes}\}} (1-P(E_{p^2})) = \prod_{p\in\{\text{primes}\}} (1-(p^2)^{-s})$$

$$= \prod_{p\in\{\text{primes}\}} (1-p^{-2s}) \overset{(b)}{=} \frac{1}{\zeta(2s)}$$

d) $H = GCD(X,Y)$. Show $P(H=n) = \frac{1}{n^{2s}\zeta(2s)}$

$$P(H=n) = P\Big(\underbrace{\{n \text{ divides } X,\ n \text{ divides } Y\}}_{A} \cap \underbrace{\Big\{\bigcap_{p\in\{\text{primes}\}}\{\{np \text{ does not divide } X\} \cup \{np \text{ does not divide } Y\}\}\Big\}}_{B}\Big) =$$

$= P(A\cap B) = P(B|A)\,P(A)$ where

$P(B|A) \overset{(c)}{=} \frac{1}{\zeta(2s)}$ and $P(A) = (n^{-s})^2$.