①

$h(n) = \{\underset{\uparrow}{1}, 1, 1\}$

a) $H(z) = \sum_{n=0}^{2} h(n)\cdot z^{-n} = 1 + z^{-1} + z^{-2}$

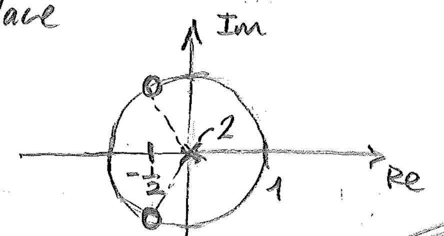
$$= \frac{z^2+z+1}{z^2} = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

Poles: $p_1 = p_2 = 0$

Zeros: $z^2+z+1 = 0$

$$z_{1,2} = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 1} = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$$

z-plane

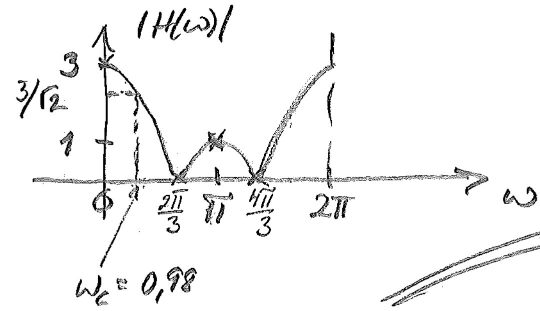


$$= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1\cdot e^{\pm j\frac{2\pi}{3}}$$

b) $H(\omega) = \sum_{n=0}^{2} h(n)\cdot e^{-j\omega n} = 1 + 1\cdot e^{-j\omega} + 1\cdot e^{-j2\omega}$

$$= e^{-j\omega}\underbrace{\left(e^{j\omega} + e^{-j\omega}\right)}_{2\cos(\omega)} + e^{-j\omega} = (1 + 2\cdot\cos(\omega))\,e^{-j\omega}$$

$|H(\omega)| = |1 + 2\cos(\omega)|$



- Max då $\cos(\omega) = 1$
  when $\omega = 0, 2\pi$
  $|H(0)| = 3$
- $|H(\omega)| = 0$
  when $\cos(\omega) = -\frac{1}{2}$
  $\Rightarrow \omega = \pm 2\pi/3$
- $|H(\pi)| = |1 + 2\cdot\cos(\pi)|$
  $= |1 + 2(-1)| = 1$

c) $|H(\omega)|_{\omega=\omega_c} = \frac{1}{\sqrt{2}}\cdot 3$

$\Rightarrow\ 1 + 2\cos(\omega_c) = \frac{3}{\sqrt{2}}$

$\cos(\omega_c) = \left(\frac{3}{\sqrt{2}} - 1\right)/2$

$\omega_c = \arccos\left(\left(\frac{3}{\sqrt{2}} - 1\right)/2\right) = 0{,}976$ rad/s

② $y(n) - y(n-1) + (0{,}25+a)\,y(n-2) = x(n-1)$

a) $\downarrow \mathcal{Z}$

$$Y(z)\left[1 - z^{-1} + (0{,}25+a)\,z^{-2}\right] = z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} + (0{,}25+a)\,z^{-2}} \quad \begin{matrix} \times z^2 \\ \times z^2 \end{matrix}$$

$$= \frac{z}{z^2 - z + (0{,}25+a)} = \frac{z}{(z-p_1)(z-p_2)}$$

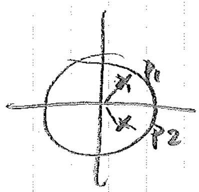
A ~~kausal~~ causal system is stable if the poles are strict inside the unit circle, i.e $|p_i| < 1$ all i.

Poles: $z^2 - z + (0{,}25+a) = 0$

$$p_{1,2} = \frac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 - (0{,}25+a)}$$

$$= \frac{1}{2} \pm \sqrt{-a} = \frac{1}{2} \pm j\sqrt{a}$$

$$= \underbrace{\sqrt{0{,}25+a}}_{|p_i|}\; e^{j\pm\theta}$$



$$|p_i| = \sqrt{0{,}25+a} < 1$$

$$\Rightarrow \; a < 0{,}75$$

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b) $y(n) - y(n-1) + 0{,}5\,y(n-2) = x(n-1)$; $x(n) = u(n)$

$\mathcal{Z} \downarrow$

$$Y(z)\left[1 - z^{-1} + 0{,}5\,z^{-2}\right] = z^{-1}U(z)$$

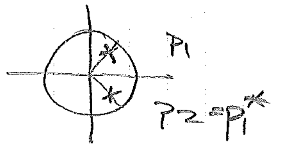
$$Y(z) = \frac{z^{-1}}{(1 - z^{-1} + 0{,}5z^{-2})} \cdot \frac{1}{(1 - z^{-1})}$$

$$= \frac{z}{(z^2 - z + 0{,}5)} \cdot \frac{z}{(z-1)}$$

$$\frac{Y(z)}{z} = \frac{z}{\underbrace{(z^2 - z + 0{,}5)}_{(z-p_1)(z-p_2)}(z-1)} = \frac{A}{(z-p_1)} + \frac{B}{(z-p_2)} + \frac{C}{(z-1)}$$

② cont.

Poles p_1 and p_2 ? :



$$z^2 - z + 0{,}5 = 0$$

$$p_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{2}} = \frac{1}{2} \pm \sqrt{-\frac{1}{4}}$$

$$= \frac{1}{2} \pm j\frac{1}{2} = \frac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

$$\begin{cases} A = (z-p_1)\dfrac{Y(z)}{z}\Big|_{z=p_1} = \dfrac{p_1}{(p_1-p_2)(p_1-1)} = \dfrac{\frac{1}{2}+j\frac{1}{2}}{j\left(-\frac{1}{2}+j\frac{1}{2}\right)} = -1 \\ B = (z-p_2)\dfrac{Y(z)}{z}\Big|_{z=p_2} = \dfrac{p_2}{(p_2-p_1)(p_2-1)} = \dfrac{\frac{1}{2}-j\frac{1}{2}}{-j\left(-\frac{1}{2}-j\frac{1}{2}\right)} = -1 \\ C = (z-1)\dfrac{Y(z)}{z}\Big|_{z=1} = \dfrac{1}{(1-p_1)(1-p_2)} = \dfrac{1}{\left(\frac{1}{2}-j\frac{1}{2}\right)\left(\frac{1}{2}+j\frac{1}{2}\right)} \end{cases}$$

$$= \frac{1}{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 2$$

$$Y(z) = \underbrace{\frac{A}{(1-p_1 z^{-1})} + \frac{A^*}{(1-p_1^* z^{-1})}}_{2|A|\cdot r_1^n \cos(\beta_1 n + \mathrm{Arg}\{A\})} + \frac{2}{(1-z^{-1})}$$

$$\Big\downarrow z^{-1}$$

$$p_1 = r_1 e^{j\beta_1} = \frac{1}{\sqrt{2}} e^{j\pi/4}$$

$$A = -1 \; ; \; |A| = 1 \; ; \; \mathrm{Arg}\{A\} = \pm\pi$$

$$y(n) = \left[2\cdot\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n - \pi\right) + 2\right] u(n)$$

$$= \left[-2\cdot\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n\right) + 2\right]\cdot u(n)$$

③

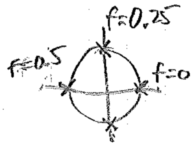
$$x(t) = \sin(2\pi \underline{F_0} t) + 0{,}33 \sin(2\pi \underline{3 \cdot F_0} \cdot t) + 0{,}2 \sin(2\pi \underline{5 \cdot F_0} \cdot t) + 0.14 \sin(2\pi \underline{7 \cdot F_0} \cdot t)$$

where $F_0 = 200\,Hz$ and $F_s = 800\,Hz$

---

a) sampling with F_s $\curvearrowright$ $\begin{cases} F \to f = \frac{F}{F_s} \ ; f \in [-0.5, 0.5] \\ t \to n \end{cases}$

$$x(n) = \sin(2\pi \underbrace{\tfrac{200}{800}}_{0.25} \cdot n) + 0{,}33 \sin(2\pi \underbrace{\tfrac{3 \cdot 200}{800}}_{0.75} \cdot n) + 0{,}2 \sin(2\pi \underbrace{\tfrac{5 \cdot 200}{800}}_{1.25} n) + 0.14 \sin(2\pi \underbrace{\tfrac{7 \cdot 200}{800}}_{1.75} \cdot n)$$



alias! $f \in [-0.5, 0.5]$

$$= \sin(2\pi \cdot 0.25\, n) + 0.33 \sin(2\pi(-0.25)n) + 0.2 \sin(2\pi(0.25) \cdot n) + 0.14 \sin(2\pi(-0.25)n)$$

$$= (1 - 0.33 + 0{,}2 - 0.14) \sin(2\pi\, 0{,}25 \cdot n)$$

$$= 0{,}73 \sin(2\pi\, 0{,}25 \cdot n)$$

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b) $h(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$

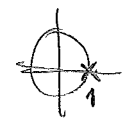
$$h(n) = u(n) - u(n-4)$$

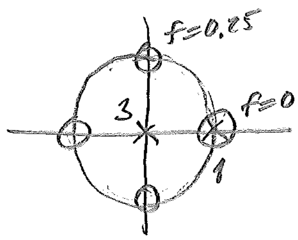
$$H(z) = U(z) - z^{-4} U(z) = \frac{1 - z^{-4}}{1 - z^{-1}} \frac{\times z^4}{\times z^4} = \frac{z^4 - 1}{z^3(z-1)}$$

poles: $\begin{cases} p_1 = p_2 = p_3 = 0 \\ p_4 = 1 \end{cases}$

zeros: $z^4 - 1 = 0 \Rightarrow z^4 = 1 \quad (z = r \cdot e^{j\theta})$

$$r^4 e^{j4\theta} = 1 \cdot e^{jk \cdot 2\pi} \ ; \ k = 0, 1, 2, 3$$





$$\begin{cases} r = 1 \\ \theta = k \cdot \frac{\pi}{2} \quad k = 0, 1, 2, 3 \end{cases}$$

$\Rightarrow$ zero at $f = 0.25$ $\Rightarrow$ $y(n) = 0$

④

a) $h(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3} \}$

$x(n) = \{ \underset{\uparrow}{1}, 1, 1, -1, -1, -1 \}$

$y(n) = h(n) * x(n)$ by graphical method:

$\boxed{1\ 1\ 1\ -1\ -1\ -1}$ length = 6

$\boxed{1/3\ 1/3\ 1/3} \longrightarrow$ length = 3

$y(n)$ $\boxed{\underset{\uparrow}{x}\ x\ x\ x\ x\ x\ \boxed{x|x}}$ length = 6+3−1 = 8

$y(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{2}{3}, 1, \tfrac{1}{3}, -\tfrac{1}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{3} \}$

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b) 6-point DFT

$\mathrm{DFT}_{⑥}\{x(n)\} \cdot \mathrm{DFT}_{⑥}\{h(n)\} \longleftrightarrow$ time domain $x(n) \, Ⓝ \, h(n)$ circ. convolution

Graphical method:

$x(n) = \{ \underset{\uparrow}{1}, 1, 1, -1, -1, -1 \}$

$h(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0 \}$ zero-padded to length N=6.

$\boxed{1\ 1\ 1\ -1\ -1\ -1}$

$\cdots \,|\, 0\ 0\ 0\ 1/3\ 1/3\ 1/3 \,|\, 0\ 0\ 0\ 1/3\ 1/3\ 1/3 \,|\, \cdots$ h(n), periodic = 6

$\cdots \,|\, \underset{\uparrow}{-1/3}\ 1/3\ 1\ 1/3\ -1/3\ -1 \,|\, \cdots$ $y_p(n)$, periodic = 6

$\hat{y}(n) = y_p(n)$; $n = 0, 1, 2, \ldots, 5^{=(N-1)}$

$\Rightarrow \hat{y}(n) = \{ \underset{\uparrow}{-\tfrac{1}{3}}, \tfrac{1}{3}, 1, \tfrac{1}{3}, -\tfrac{1}{3}, -1 \}$

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(4) cont.

c) $F_1 = 1200\ Hz$ $\quad$ $F_s = 10\ kHz$ $\quad$ $f_1 = \frac{1200}{10k} = \frac{12}{100}$

$F_2 = 4200\ Hz$ $\quad$ $f = \frac{F}{F_s}$ $\quad$ $f_2 = \frac{4200}{10k} = \frac{42}{100}$

$F_3 = 6800\ Hz$ $\quad$ $f_3 = \frac{6800}{10k} = \frac{68}{100}$

Alias problem for f_3!

$$f_3 = \frac{68}{100} = -\frac{32}{100} + 1 \Rightarrow \hat{f}_3 = -\frac{32}{100}$$

(F_3 is seen as 3200 Hz after sampling)

From figure: and $\begin{cases} f_k = \frac{k}{N} & ; k = 0, 1, 2, \dots 1023 \\ F_k = f_k \cdot F_s \end{cases}$

$$\Rightarrow F_{123} = \frac{123}{1024} \cdot 10k \approx 1200 \xrightarrow{\text{Identify}} k = 123 \rightarrow F_1$$

$$F_{328} = \frac{328}{1024} \cdot 10k \approx 3200 \longrightarrow k = 328 \rightarrow \text{alias version of } F_3$$

$$F_{430} = \frac{430}{1024} \cdot 10k = 4200 \longrightarrow k = 430 \rightarrow F_2$$

$F_{594}, F_{696}, F_{901}$ are the "neg. frequencies".