Solutions Multivariable calculus, 2010-10-29.

1. Find an equation of the tangent plane to the surface $x^3 - zy^2 + 2z^3 + 1 = 0$ at the point $(1, 2, 1)$. *(2p)*

   Answer:
   Setting $F(x, y, z) = x^3 - zy^2 + 2z^3 + 1$, the surface above can be viewed as the level surface $F(x, y, z) = 0$. A normal vector, $\boldsymbol{n}$, to the tangent plane is then:

   $$\boldsymbol{n} = \nabla F(x, y, z)_{(1,2,1)} = (3x^2, -2zy, -y^2 + 6z^2)_{(1,2,1)} = (3, -4, 2).$$

   This gives the equation of the tangent plane:

   $$\boldsymbol{n} \cdot (x - 1, y - 2, z - 1) = 0 \Leftrightarrow 3\,(x - 1) + (-4)\,(y - 2) + 2\,(z - 1) = 0 \Leftrightarrow 3x - 4y + 2z + 3 = 0.$$

2. Find the following limit or show that it does not exist $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$. *(2p)*

   Answer:
   Set : $x = r\cos\theta$, $y = r\sin\theta$. This gives :

   $$\frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \frac{r^2(\cos^2\theta - \sin^2\theta)}{r} = r\,\cos 2\theta\,.$$

   Now, $|\cos 2\theta| \le 1$, and $r \to 0$ as $(x, y) \to (0, 0)$. Therefore: $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}} = 0.$

3. Find and classify all critical points of the function $f(x, y) = x^2 + y^2 - 6xy + 5x + 9y$. *(2p)*

   Answer:
   We first find all critical points (as it turns out, there is only one):

   $$\begin{array}{rcl} f_x &=& 0 \\ f_y &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2x - 6y + 5 &=& 0 \\ -6x + 2y + 9 &=& 0 \end{array} \Leftrightarrow (x, y) = (2, \tfrac{3}{2}).$$

   Also, we need the 2nd partial derivatives: $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = -6$.
   An expansion to 2nd order around the point $(2, \frac{3}{2})$ then yields:

   $$f(2+h, \tfrac{3}{2}+k) \simeq f(2, \tfrac{3}{2}) + \tfrac{1}{2}\Big(2h^2 + 2\cdot(-6)\cdot hk + 2k^2\Big) = \tfrac{47}{4} + h^2 - 6hk + k^2 = \tfrac{47}{4} + (h - 3k)^2 - 8k^2.$$

   From this expression we see that that $(2, \frac{3}{2})$ is a saddle-point.

4. Calculate $\iint\limits_D x^2\, e^{xy}\; dx\, dy$, where D is a triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$. *(3p)*

   Answer:

   $$\iint\limits_D x^2\, e^{xy}\, dx\, dy = \int_0^1 x^2 \Big(\int_0^x e^{xy}\, dy\Big)\, dx = \int_0^1 x^2 \Big[\frac{1}{x}\, e^{xy}\Big]_0^x\, dx = \int_0^1 x\,(e^{x^2} - 1)\, dx =$$

   $$= \Big[\frac{1}{2}e^{x^2} - \frac{1}{2}x^2\Big]_0^1 = \frac{1}{2}\,(e - 2).$$

5. Calculate $\displaystyle\iint_{\Delta} \frac{(2x+y)^2}{(2x-y)^2+1}\,dx\,dy$,

where $\Delta = \{(x,y) \in \mathbb{R}^2 \mid 0 \le 2x-y \le 1,\ 0 \le 2x+y \le 2\}$. *(3p)*

Answer:
The parallellogram Δ in the $xy-$plane can be transformed into a rectangle $\Delta_{uv}:\ 0 \le u \le 1,\ 0 \le v \le 2$ by the substitution:

$u = 2x-y,\ v = 2x+y,$ which gives $x = \frac{1}{4}u + \frac{1}{4}v,\ y = -\frac{1}{2}u + \frac{1}{2}v,$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{4}.$

Inserted into the integral:

$$\iint_{\Delta} \frac{(2x+y)^2}{(2x-y)^2+1}\,dx\,dy = \int_0^2\int_0^1 \frac{v^2}{u^2+1}\left|\frac{1}{4}\right| du\,dv = \frac{1}{4}\int_0^2 v^2\,dv \int_0^1 \frac{1}{u^2+1}\,du =$$
$$\frac{1}{4}\left[\frac{1}{3}v^3\right]_0^2 \left[\arctan u\right]_0^1 = \frac{\pi}{6}.$$

6. Find the absolute minimum and maximum values of $f(x,y,z) = 3x - 2y - z$ subject to the constraint $x^2+y^2+z^2 = 14$. *(3p)*

Answer:
Using the method of Lagrange multipliers and letting $g(x,y,z) = x^2+y^2+z^2-14 = 0$ represent the constraint :

$$\begin{array}{rcl} f_x &=& \lambda g_x \\ f_y &=& \lambda g_y \\ f_z &=& \lambda g_z \end{array} \Leftrightarrow \begin{array}{rcl} 3 &=& 2\lambda x \\ -2 &=& 2\lambda y \\ -1 &=& 2\lambda z \end{array} \Rightarrow \begin{array}{rcl} 2\lambda x &=& (-3)\cdot 2\lambda z \\ 2\lambda y &=& 2\cdot 2\lambda z \end{array} \Rightarrow (x,y,z) = (-3z, 2z, z).$$

This solution is then inserted into the constraint (note that $\lambda = 0$ is inconsistent with the original Lagrange equations):

$$(-3z)^2 + (2z)^2 + z^2 = 14 \Leftrightarrow z = -1 \vee z = 1.$$

Comparison and conclusion:

$f(3,-2,-1) = 14,\ f(-3,2,1) = 14,\ f_{min} = -14,\ f_{max} = 14.$

7. Find the absolute minimum and maximum values of $g(x,y) = (x^2+y^2)\,e^{-x^2-4y^2}$ on the set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+4y^2 \le 4\}$. *(5p)*

Answer:
First finding interior critical points:

$$\begin{array}{rcl} g_x &=& 0 \\ g_y &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2xe^{-x^2-4y^2}(1-x^2-y^2) &=& 0 \\ 2ye^{-x^2-4y^2}(1-4x^2-4y^2) &=& 0 \end{array} \Leftrightarrow$$
$(x=0 \vee x^2+y^2=1) \wedge (y=0 \vee 4x^2+4y^2=1) \Leftrightarrow (x,y)=(0,0) \vee (x,y)=(0,\pm\frac{1}{2}) \vee (x,y)=(\pm 1,0).$

All of these 5 points are interior to A.

Next we consider the values of g on the boundary of A, that is, the ellipse $x^2+4y^2=4$.
We parametrize the boundary curve as: $x = 2\cos t,\ y = \sin t,\ 0 \le t < 2\pi$. Inserted:

$$g(2\cos t, \sin t) = (4\cos^2 t + \sin^2 t)\,e^{-4} = (3\cos^2 t + 1)\,e^{-4}.$$

From the expression above, the maximum and minimum values *on the boundary* are seen to be $g(2,0) = g(-2,0) = 4e^{-4}$ and $g(0,1) = g(0,-1) = e^{-4}$, respectively.

Comparing with the values at the interior critical points:

$g(0,0) = 0,\ g(0,\pm\frac{1}{2}) = \frac{e^{-1}}{4},\ g(\pm1,0) = e^{-1}.$

Since $e^{-1} > 4e^{-4}$, we then have the following absolute extrema on A:

$g_{min} = 0,\ g_{max} = e^{-1}.$

8. Calculate $\displaystyle\iiint\limits_K \frac{xyz}{x^2+y^2+z^2+1}\,dx\,dy\,dz$,

$K = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$. *(5p)*

Answer:
Using spherical coordinates and noting that K is the 1st octant part of a sphere with radius $R = 1$ we obtain the limits:

$0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 1.$

The integral can then be calculated as:

$$\iiint\limits_K \frac{xyz}{x^2+y^2+z^2+1}\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{\rho\sin\phi\cos\theta\cdot\rho\sin\phi\sin\theta\cdot\rho\cos\phi}{\rho^2+1}\,\rho^2 d\rho\ \sin\phi\,d\phi\,d\theta =$$

$$\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta \int_0^{\frac{\pi}{2}} (1-\cos^2\phi)\,\cos\phi\,\sin\phi\,d\phi \int_0^1 \frac{\rho^5}{\rho^2+1}\,d\rho = \Big[u = \cos\phi\Big] =$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta \int_0^1 (1-u^2)\,u\,du \int_0^1 \Big(\rho^3 - \rho + \frac{\rho}{\rho^2+1}\Big)\,d\rho =$$

$$\frac{1}{2}\Big[-\frac{1}{2}\cos(2\theta)\Big]_0^{\frac{\pi}{2}} \Big[\frac{1}{2}u^2 - \frac{1}{4}u^4\Big]_0^1 \Big[\frac{1}{4}\rho^4 - \frac{1}{2}\rho^2 + \frac{1}{2}\ln(\rho^2+1)\Big]_0^1 = \frac{1}{32}\,(2\ln 2 - 1).$$

Alternatively, the integral can be calculated as (where K_{xy} is the projektion of K on to the xy-plane):

$$\iiint\limits_K \frac{xyz}{x^2+y^2+z^2+1}\,dx\,dy\,dz = \int\limits_{K_{xy}} xy\,\Big(\int_0^{\sqrt{1-x^2-y^2}} \frac{z}{x^2+y^2+z^2+1}\,dz\Big)\,dx\,dy =$$

$$\iint\limits_{K_{xy}} xy\,\Big[\frac{1}{2}\ln(x^2+y^2+z^2+1)\Big]_{z=0}^{\sqrt{1-x^2-y^2}} dx\,dy = \frac{1}{2}\iint\limits_{K_{xy}} xy\,\Big(\ln 2 - \ln(x^2+y^2+1)\Big)\,dx\,dy =$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^1 r\sin\theta\,r\cos\theta\,\Big(\ln 2 - \ln(r^2+1)\Big)\,r\,dr\,d\theta =$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} \sin(2\theta)d\theta\Big(\ln 2\int_0^1 r^3 dr - \int_0^1 r^3\ln(r^2+1)dr\Big) = \frac{\ln 2}{16} - \frac{1}{4}\Big(\Big[\frac{1}{4}r^4\ln(r^2+1)\Big]_0^1 - \frac{1}{4}\int_0^1 r^4\,\frac{2r}{r^2+1}dr\Big) =$$

$$\frac{\ln 2}{16} - \frac{\ln 2}{16} + \frac{1}{8}\Big[\frac{1}{4}r^4 - \frac{1}{2}r^2 + \frac{1}{2}\ln(r^2+1)\Big]_0^1 = \frac{1}{32}\,(2\ln 2 - 1).$$

(The integral $\int r^3\ \ln(r^2+1)\,dr$ is carried out by a combination of *integration by parts* and the result above for $\int \frac{\rho^5}{\rho^2+1}\,d\rho$).

9. Calculate $\iint\limits_S z\,dS$, where the surface S is the part of the paraboloid $z = 2 - x^2 - y^2$ that lies *above* the cone $z = \sqrt{x^2+y^2}$. *(5p)*

Answer:
We first note that (with $z = 2 - x^2 - y^2$): $\sqrt{1 + z_x^2 + z_y^2} \;=\; \sqrt{1 + 4x^2 + 4y^2}$.

Furthermore, the projection of the surface S on to the xy - plane is given by the condition

$$2 - x^2 - y^2 \geq \sqrt{x^2+y^2} \;\Leftrightarrow\; r^2 + r - 2 \leq 0 \;\Rightarrow\; r \leq 1, \quad (r = \sqrt{x^2+y^2}).$$

This means that the projection of S on to the xy - plane is a unit disk centered at the origin. The integral is then calculated as:

$$\iint\limits_S z\,dS \;=\; \iint\limits_{S_{xy}} (2 - x^2 - y^2)\,\sqrt{1 + 4x^2 + 4y^2}\;dx\,dy \;=$$

$$\int_0^{2\pi}\int_0^1 (2 - r^2)\,\sqrt{1 + 4r^2}\;r\,dr\,d\theta \;=\; \left[u = 1 + 4r^2\right] \;=$$

$$\frac{\pi}{16}\int_1^5 (9 - u)\,\sqrt{u}\,du \;=\; \frac{\pi}{16}\left[6u^{\frac{3}{2}} \;-\; \frac{2}{5}\,u^{\frac{5}{2}}\right]_1^5 \;=\; \frac{\pi}{20}\,(25\sqrt{5} - 7).$$