Optimal disturbance rejection and tracking

Outline

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1. Optimal disturbance rejection
   - Signal models
   - Annihilation
2. Optimal reference tracking
   - Designing T for tracking
   - Robust tracking
   - Pre-view action design
3. Examples
   - Repetitive design
   - Pre-view action design

Signal models

Pulse response model

$$d(k) = \frac{N(\mathrm{q}^{-1})}{D(\mathrm{q}^{-1})}\delta(k)$$

Signal models

Pulse response model

$$d(k) = \frac{N(\mathrm{q}^{-1})}{D(\mathrm{q}^{-1})}\delta(k)$$

Example

Step $d(k) = a, k \geq 0$

$$d(k) = \frac{a}{1-\mathrm{q}^{-1}}\delta(k), \quad \begin{cases} d(k) = d(k-1) + a\delta(k) \\ d(0) = 0 + a \cdot 1 = a \\ d(1) = a + a \cdot 0 = a \\ \vdots \\ d(k) = a, k \geq 0 \end{cases}$$

Ramp signal

$d(k) = a + bk, \quad k \geq 0$

$$d(k) = \frac{n_0 + n_1 \mathrm{q}^{-1}}{(1-\mathrm{q}^{-1})^2}\delta(k), \quad \begin{cases} n_0 = a \\ n_1 = b - a \end{cases}$$

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check $d(k) = 2d(k-1) - d(k-2) + n_0\delta(k) + n_1\delta(k-1)$

$$\begin{aligned}
&d(0) = n_0 \\
&d(1) = 2[n_0] + n_1 = n_0 + (n_0 + n_1) \\
&d(2) = 2[n_0 + (n_0 + n_1)] - n_0 = n_0 + 2(n_0 + n_1) \\
&d(3) = 2[n_0 + 2(n_0 + n_1)] - [n_0 + (n_0 + n_1)] = n_0 + 3(n_0 + n_1)
\end{aligned}$$

Sinusoidal signal

$d(k) = a\cos(\omega k + b),\ k \geq 0$

$$d(k) = \frac{n_0 + n_1 \mathrm{q}^{-1}}{1 - 2\cos\omega \mathrm{q}^{-1} + \mathrm{q}^{-2}}\delta(k), \quad \left\{ \begin{array}{l} n_0 = a\cos b \\ n_1 = -a\cos(b-\omega) \end{array} \right.$$

Sinusoidal signal

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sketch of proof

Complex $d^*(k) = ae^{i(\omega k + b)}$ where $d(k) = \Re[d^*(k)]$

$$\begin{aligned} d^*(k) &= \frac{ae^{ib}}{1-e^{i\omega}\mathrm{q}^{-1}}\delta(k) \\ &= \frac{ae^{ib}(1-e^{-i\omega}\mathrm{q}^{-1})}{(1-e^{i\omega}\mathrm{q}^{-1})(1-e^{-i\omega}\mathrm{q}^{-1})}\delta(k) \\ &= \frac{ae^{ib}-ae^{i(b-\omega)}\mathrm{q}^{-1}}{1-2\cos\omega\mathrm{q}^{-1}+\mathrm{q}^{-2}}\delta(k) \end{aligned}$$

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Example

For $d(k) =$

$$\begin{array}{ll} \cos\omega k, & n_0 = 1,\, n_1 = -\cos\omega \\ \sin\omega k, & n_0 = 0,\, n_1 = \sin\omega \end{array}$$

Periodic signal

$d(k) = d(k-T),\ k \geq T$

$$d(k) = \frac{n_0 + \cdots + n_{T-1} q^{-(T-1)}}{1 - q^{-T}} \delta(k), \quad n_k = d(k), 0 \leq k \leq T-1$$

Periodic signal

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check $d(k) = d(k-T) + n_0\delta(k) + \ldots + n_{T-1}\delta(k-(T-1))$

$$\begin{aligned} d(0) &= n_0 \\ d(1) &= n_1 \\ &\vdots \\ d(T-1) &= n_{T-1} \\ d(T) &= d(0) = n_0 \\ d(k) &= d(k-T), k \geq T \end{aligned}$$

Signal model classes

$d(k)$	$D(\mathrm{q}^{-1})$
a	$1-\mathrm{q}^{-1}$
$a+bk$	$(1-\mathrm{q}^{-1})^2$
$a+bk+ck^2$	$(1-\mathrm{q}^{-1})^3$
$a\cos(\omega k+b)$	$1-2\cos\omega\mathrm{q}^{-1}+\mathrm{q}^{-2}$
$d(k)=d(k-T)$	$1-q^{-T}$

Signal classes defined by $D(\mathrm{q}^{-1})$ independent of a, b and c

Particular signal in model class

Signal in class defined by $N(\mathrm{q}^{-1})$ (dependent on a, b and c)

$$
\begin{aligned}
&D(\mathrm{q}^{-1})d(k) = N(\mathrm{q}^{-1})\delta(k)\\
&d(0) = n_0\\
&d(1) + d_1 d(0) = n_1\\
&d(2) + d_1 d(1) + d_2 d(0) = n_2\\
&\vdots\\
&d(\deg D) + d_1 d(\deg D - 1) + \ldots + d_{\deg D} d(0) = n_{\deg D} = 0
\end{aligned}
$$

Notice that since $\deg N < \deg D$

$$D(\mathrm{q}^{-1})d(k) = 0, \quad k \geq \deg D$$

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Annihilation polynomial

Annihilation polynomial D

$$D(\mathrm{q}^{-1})d(k) = 0, \quad k \geq \deg D$$

Annihilation polynomial

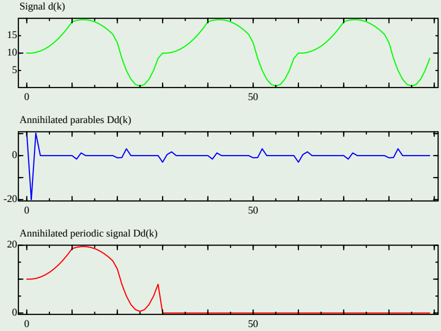
Annihilation polynomial D

$$D(\mathrm{q}^{-1})d(k) = 0, \quad k \geq \deg D$$

Example

$D(\mathrm{q}^{-1}) =$

- $(1 - \mathrm{q}^{-1})^3$
- $1 - q^{-30}$



Annihilation principle

Process and controller

$$\begin{aligned} Ay &= Bu + Cd \\ Ru &= -Sy + Tr \end{aligned}$$

Closed-loop dynamics

$$\begin{aligned} y &= \frac{BT}{A_c} r + \frac{RC}{A_c} d \\ u &= \frac{AT}{A_c} r - \frac{SC}{A_c} d \end{aligned}$$

Annihilation principle

Process and controller

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Closed-loop dynamics

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\end{aligned}
$$

Annihilation of disturbance $R = R_1 D$

$$
\frac{RC}{A_c} d = \frac{R_1 D C}{A_c} \frac{N}{D} \delta = \frac{R_1 CN}{A_c} \delta(k) \to 0, k \to \infty
$$

Robustness considerations

Observations:

- Signal model has poles at the unit circle (otherwise $d \longrightarrow 0$)
- $R_f = D$ blows up Nyquist curve to infinity at these poles
- Robust design therefore very important!

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An introductory example

Process

$$y = \frac{\mathrm{q}^{-1} + \mathrm{q}^{-2}}{(1 - \lambda_1 \mathrm{q}^{-1})(1 - \lambda_2 \mathrm{q}^{-1})} u, \quad \lambda_{1,2} = 0.8 \pm 0.2i$$

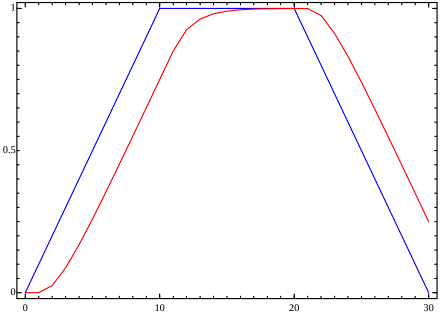
Compare designs with $A_c = 1 - 0.5\mathrm{q}^{-1}$
Piecewise ramps reference

$$D(\mathrm{q}^{-1}) r(k) = 0$$

with annihilator $D = (1 - \mathrm{q}^{-1})^2$

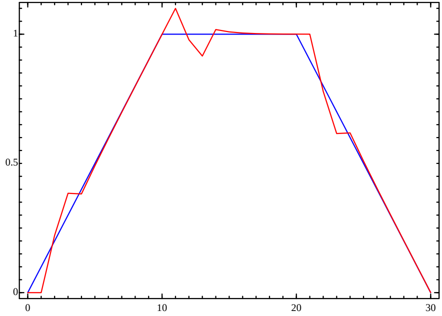
T adjusted for steady-state gain 1

Design 1: $T = \frac{A_c(1)}{B(1)}$



$R_f = D$ and $T = S$

Design 2: Fix factor $R_f = D$ and error feedback $T = S$



Analysis

Tracking error

$$e(k) = r(k) - y(k) = [1 - \frac{BT}{A_c}]r(k) = \frac{A_c - BT}{A_c}\frac{N}{D}\delta(k)$$

Design 1 $[A_c - BT]$ includes factor $1 - \mathrm{q}^{-1}$: Tracks steps but not ramps

Design 2 $[A_c - BS] = AR = AR_1 D$: Tracks ramps

T design for tracking signal class D

Include factor D in $[A_c - BT]$ by solving polynomial equation

$$BT + DM = A_c \rightarrow \begin{cases} T \\ M \end{cases}$$

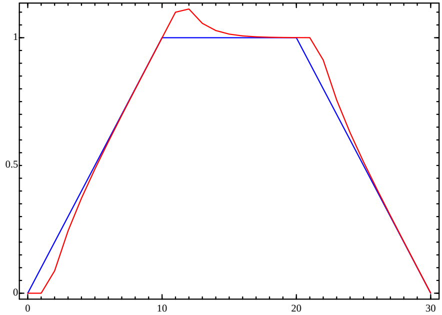
Tracking error

$$e(k) = \frac{A_c - BT}{A_c}\frac{N}{D}\delta(k) = \frac{MN}{A_c}\delta(k) \rightarrow 0, k \rightarrow \infty$$

if A_c is stable

T design for tracking ramps without fix factor

Design 3: $R_f = 1$ and T from $BT + DM = A_c$



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Model mismatch regarded as disturbance

Process, controller and closed-loop dynamics

$$\begin{aligned} A^* y &= B^* u \\ Ru &= -Sy + Tr \end{aligned} \qquad \begin{aligned} y &= \frac{B^* T}{A_c^*} r \\ u &= \frac{A^* T}{A_c^*} r \end{aligned} \qquad r = \frac{N}{D}\delta$$

Model mismatch regarded as disturbance

Process, controller and closed-loop dynamics

$$A^* y = B^* u \qquad y = \frac{B^* T}{A_c^*} r \qquad r = \frac{N}{D}\delta$$
$$Ru = -Sy + Tr \qquad u = \frac{A^* T}{A_c^*} r$$

Model and closed-loop dynamics

$$Ay = Bu + \varepsilon \qquad y = \frac{BT}{A_c} r + \frac{R}{A_c}\varepsilon$$

Model mismatch regarded as disturbance

Process, controller and closed-loop dynamics

$$\begin{aligned} A^* y &= B^* u \\ Ru &= -Sy + Tr \end{aligned} \qquad \begin{aligned} y &= \frac{B^* T}{A_c^*} r \\ u &= \frac{A^* T}{A_c^*} r \end{aligned} \qquad r = \frac{N}{D}\delta$$

Model and closed-loop dynamics

$$Ay = Bu + \varepsilon \qquad y = \frac{BT}{A_c} r + \frac{R}{A_c}\varepsilon$$

Model mismatch as disturbance

$$\begin{aligned} \varepsilon \quad & = Ay - Bu = (A - A^*)y - (B - B^*)u \\ & = \frac{[(A-A^*)B^*T-(B-B^*)A^*T]}{A_c^*} r = \frac{C}{A_c^*} r = \frac{C}{A_c^*}\frac{N}{D}\delta \\ D\varepsilon(k) \quad & \rightarrow 0, k \rightarrow \infty \qquad \text{(if } A_c^* \text{ stable)} \end{aligned}$$

Tracking despite model mismatch

Polynomial equations

$$\begin{cases} ADR_1 + BS = A_c & \rightarrow R_1, S \quad (R = DR_1) \\ BT + DM = A_c & \rightarrow T, M \end{cases}$$

Tracking despite model mismatch

Polynomial equations

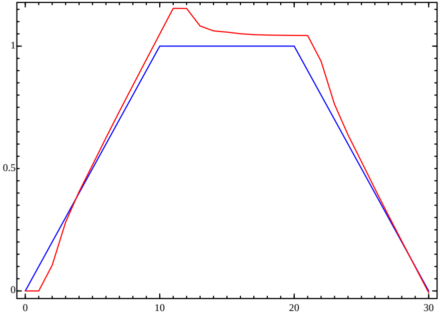
$$\left\{ \begin{array}{ll} ADR_1 + BS = A_c & \rightarrow R_1, S \quad (R = DR_1) \\ BT + DM = A_c & \rightarrow T, M \end{array} \right.$$

Tracking error

$$\begin{aligned} e &= r - y = r - [\tfrac{BT}{A_c} r + \tfrac{R}{A_c} \varepsilon] \\ &= \tfrac{[A_c - BT]}{A_c} \tfrac{N}{D} \delta - \tfrac{RCN}{A_c A_c^* D} \delta \\ &= \tfrac{MN}{A_c} \delta(k) + \tfrac{R_1 CN}{A_c A_c^*} \delta(k) \rightarrow 0, k \rightarrow \infty \end{aligned}$$

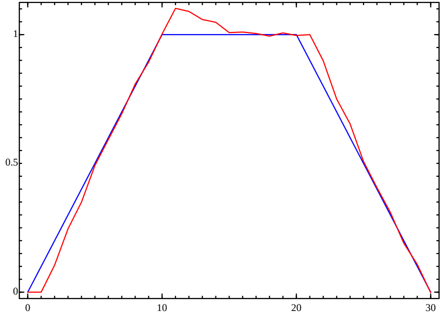
T design for tracking ramps without fix factor

Design 3 with modeling error $B^* = 1.2B$ ($A^* = A$)



T design for tracking ramps with fix factor

Design 4: robust tracking (two polynomial equations)



Outline

Control signal in advance

Process with delay τ

$$y(k) = \frac{B}{A}u(k) = \frac{b_\tau q^{-\tau} + \dots}{1 + a_1 \mathrm{q}^{-1} + \dots}u(k) = \frac{B_\tau}{A}q^{-\tau}u(k)$$

If r known, introduce control signal in advance

$$Ru(k) = -Sy(k) + Tr(k + \tau_p)$$

Closed-loop dynamics

$$y = \frac{TB}{A_c}r(k + \tau_p) = \frac{TB_\tau q^{-\tau}}{A_c}q^{\tau_p}r = \frac{TB_\tau}{A_c q^{-(\tau_p - \tau)}}r$$

Pre-view action design

Polynomial equation for pre-view design

$$B_\tau T + DM = A_c q^{-(\tau_p - \tau)} \longrightarrow T, M$$

Pre-view action design

Polynomial equation for pre-view design

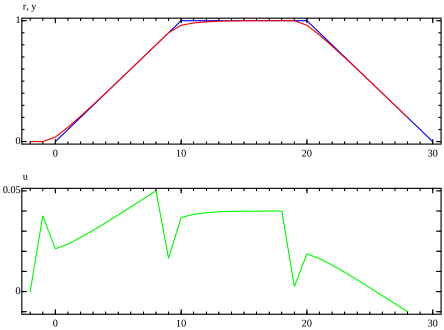
$$B_\tau T + DM = A_c q^{-(\tau_p - \tau)} \rightarrow T, M$$

Tracking error

$$\begin{aligned} e(k) &= r - y = r - \frac{TB_\tau}{A_c q^{-(\tau_p-\tau)}} r \\ &= \frac{A_c q^{-(\tau_p-\tau)} - TB_\tau}{A_c q^{-(\tau_p-\tau)}} \frac{N}{D}\delta = \frac{MN}{A_c q^{-(\tau_p-\tau)}}\delta \\ &= \frac{MN}{A_c}\delta(k+\tau_p-\tau) \rightarrow 0, k \rightarrow \infty \end{aligned}$$

Previous example with pre-view action

Design with $\tau_p = 2$



Outline

Servo process

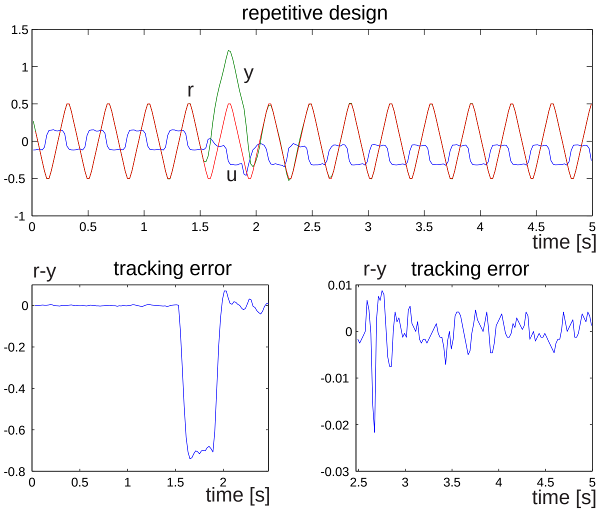
Model

$$y(k) = y(k-1) + u(k-1)$$

Repetitive design

- Triangular-like reference of period 18 samples
- Fix factor design $R_f = D = 1 - q^{-18}$
- Study convergence after step disturbance

Repetitive design



Outline

Pre-view action designs for sawtooth tracking

Feedback design

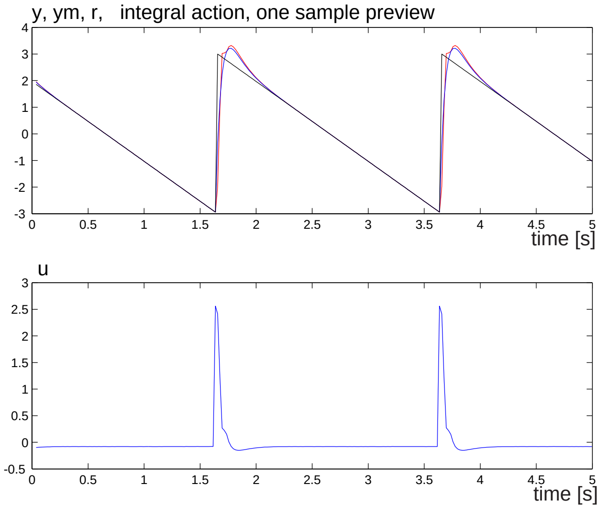
- Integral action $R_f = 1 - \mathrm{q}^{-1}$
- Pole placement $A_c = (1 - 0.7\mathrm{q}^{-1})(1 - 0.8\mathrm{q}^{-1})$

Pre-view action design

$$B_1 T + DM = A_c q^{-(\tau_p - \tau)} \longrightarrow T, M$$

- Ramp tracking $D = (1 - \mathrm{q}^{-1})^2$
- Choose deg $T = 1$ (minimal)
- Study designs for $\tau_p = \tau = 1$ and $\tau_p = 3$

Design with preview $\tau_p = \tau = 1$



Design with preview $\tau_p = 3$