Solutions to Exam for Stochastic models in Discrete time 3.75 ECTS

Master's program of Financial Mathematics
October 30, 2009, 9.00 – 13.00

Max number of points: 30.
Halmstad University grading bounds: 12p ⇒ grade 3, 18p⇒ grade 4, 24p ⇒ grade 5.
ECTS bounds: 12p ⇒ grade E, 15p ⇒ grade D, 18p ⇒ grade C, 21p ⇒ grade B, 24p ⇒ grade A.
Allowed aids: Summary of formulae attached to the exam, calculator and dictionary.
Examiner: Eric Järpe (035-16 76 53, 0702-822 844).

1. Let $X = \{(X_n, \mathcal{F}_n) : n \geq 0\}$ be a stochastic sequence with $E(|X_0|) < \infty$. Prove that if X is a martingale transformation, then X is a local martingale. (4p)

   Solution: (See pp 98 in *Essentials of Stochastic Finance. Facts, Models, Theory.* by A.N. Shiryaev.) □

2. Assume $\{X_n\}$ is an $MA(2)$ process with parameters $b_0 = b_1 = b_2 = \sigma_\epsilon^2 = 1$. Calculate

   (a) the variance $D(X_n)$ (3p)

   (b) the probability $P(X_n > X_{n+1} + 1)$ (4p)

   Solution:

   (a) $D(X_n) = D(1 + \epsilon_{n-1} + \epsilon_{n-2} + \epsilon_n) = D(\epsilon_{n-1}) + D(\epsilon_{n-2}) + D(\epsilon_n) = 3.$

   (b) $P(X_n > X_{n+1} + 1) = 1 - P(X_n - X_{n+1} \leq 1)$ where $X_n - X_{n+1}$ is normally distributed with expectation $\mu = E(X_n - X_{n+1}) = 0$ and variance $\sigma^2 = = D(X_n - X_{n+1}) = D(1 + \epsilon_n + \epsilon_{n-1} + \epsilon_{n-2} - (1 + \epsilon_{n+1} + \epsilon_n + \epsilon_{n-1})) = = D(\epsilon_{n-2} - \epsilon_{n+1}) = 2$. Thus $P(X_n > X_{n+1} + 1) = 1 - \Phi(\frac{1-0}{\sqrt{2}}) = = 1 - \Phi(0.7071) = 0.2389.$ □

3. Let $\{\xi_k\}_{k=0}^{\infty}$ be a classical random walk starting in 0 at time 0.

   (a) Prove that $\{\xi_k\}$ is non-stationary. (4p)

   (b) Calculate the probability that the random walk returns to zero at time n. (5p)

   Solution:

   (a) If the process is not weakly stationary it can not be stationary in any sense. To be weakly stationary we have to have $E(\xi_k) = m$ and $C(\xi_k, \xi_{k+h}) = R(h)$. Let us write $\xi_k = \sum_{i=1}^{k} X_i$ where $\{X_i\}$ are independent and distributed $P(X_i = -1) = P(X_i = 1) = \frac{1}{2}$. Since $E(X_i) = 0$ we have that

$D(X_i) = E(X_i^2) = (-1)^2 P(X_i = -1) + 1^2 P(X_i = 1) = 1$ and $C(\xi_k, \xi_{k+1}) = C(\sum_{i=1}^k X_i, \sum_{i=1}^{k+h} X_i) = C(\sum_{i=1}^k X_i, \sum_{i=1}^k X_i) + C(\sum_{i=1}^k X_i, \sum_{i=k+1}^{k+h} X_i) = \sum_{i=1}^k D(X_i) + 0 = k$. Thus the covariance function of $\{\xi_k\}$ can not be a function of h (the time distance between the variables), and consequently $\{\xi_k\}$ can not be stationary.

(b) Since the random walk is $\xi_n = \sum_{i=1}^n X_i$ where $\{X_i\}$ are independent and distributed $P(X_i = -1) = P(X_i = 1) = \frac{1}{2}$, the variables X_i can be written as $2Y_i - 1$ where $\{Y_i\}$ are independent and Bernoulli distributed with $\frac{1}{2}$, i.e. $P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}$ and $\sum_{i=1}^n Y_i \in Bin(n, \frac{1}{2})$. Now, $P(\xi_n = 0) = P(\sum_{i=1}^n (2Y_i - 1) = 0) = P\Big(2(\sum_{i=1}^n Y_i) - n = 0\Big) = P(\sum_{i=1}^n Y_i = \frac{n}{2}) =$
$= \begin{cases} \binom{n}{n/2}(\frac{1}{2})^{n/2}(1-\frac{1}{2})^{n-n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} = \begin{cases} \binom{n}{n/2}2^{-n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ □

4. (*Cusum procedure*) Let $\{X_k\}$ be a sequence of independent identially distributed random variables with zero mean, and let the sequence $\{S_n\}$ be defined by

$$S_n = \max_{k=1,2,\ldots,n} \sum_{j=k}^n X_j$$

(a) Derive a recursive representation of $\{S_n\}$, i.e. find a relationship f such that, with $S_1 = X_1$, we have $S_n = f(X_n, S_{n-1})$ for $n = 2, 3, \ldots$ (3p)

(b) Prove that $\{S_n\}$ is a submartingale with respect to the flow $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. (2p)

Solution:

(a)
$$\begin{aligned} S_n &= \max_{k=1,2,\ldots,n} \sum_{j=k}^n X_j \\ &= \max\left(\sum_{j=1}^n X_j, \sum_{j=2}^n X_j, \ldots, \sum_{j=n-1}^n X_j, X_n\right) \\ &= \max\left(X_n + \sum_{j=1}^{n-1} X_j, X_n + \sum_{j=2}^{n-1} X_j, \ldots, X_n + X_{n-1}, X_n + 0\right) \\ &= X_n + \max\left(\sum_{j=1}^{n-1} X_j, \sum_{j=2}^{n-1} X_j, \ldots, X_{n-1}, 0\right) \\ &= X_n + \max\Big(0, \max((\sum_{j=1}^{n-1} X_j, \sum_{j=2}^{n-1} X_j, \ldots, X_{n-1})\Big) \\ &= X_n + \max(0, S_{n-1}) \end{aligned}$$

(b) Assuming that $E(|X_n|)$ is finite $= m$, we have that $E(|S_n|) = E(|\max_{1\le k\le n} \sum_{j=k}^n X_j|) \le E(\max_{1\le k\le n} \sum_{j=k}^n |X_j|) \le E(\sum_{j=1}^n |X_j|) = mn < \infty$. Of course $S_n = \max_{1\le k\le n} \sum_{j=k}^n X_j \in \mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Finally, since $\{X_k\}$ are independent and have zero mean, we have that $E(X_{n+1}|\mathcal{F}_n) = 0$, and therefore that $E(S_{n+1}|\mathcal{F}_n) = E(X_{n+1} + \max(0, S_n)|\mathcal{F}_n) = E(X_{n+1}|\mathcal{F}_n) + E(\max(0, S_n)|\mathcal{F}_n) = 0 + \max(0, S_n) \ge S_n$. □

5. Assume that $\{h_t\}$ is a stochastic volatility process of order 1 with parameters a_0, a_1 such that $0 < a_1 < 1$ and $c = 1$. Prove that the fourth moment of h_t is bounded from above by $3e^2$. (5p)

Solution: In the stochastic volatility model of order 1 with $c = 1$ we have $h_t = \sigma_t \epsilon_t$ where $\sigma_t^2 = e^{\Delta_t}$ and $\Delta_t = a_0 + a_1 \Delta_{t-1} + \delta_t$. Then, due to stationarity $E(h_t^4) = m_4$ where

$$\begin{aligned} m_4 &= E(\sigma_t^4 \epsilon_t^4) \\ &= E(\sigma_t^4)E(\epsilon_t^4) \\ &= 3E(e^{2\Delta_t}) \\ &= 3E(e^{2(a_0+a_1\Delta_{t-1}+\delta_t)}) \\ &= 3e^{2a_0}E(e^{2a_1\Delta_{t-1}})E(e^{2\delta_t}) \end{aligned} \tag{1}$$

Now let $Y_t = e^{2\Delta_t}$. Then $m_4 = 3E(Y_t)$. Further, $f(y) = y^{a_1}$ is a concave function since $0 < a_1 < 1$. Thus $E(f(Y_{t-1})) = E(e^{2a_1\Delta_{t-1}}) \leq (E(e^{2\Delta_{t-1}}))^{a_1} = (E(f(Y_{t-1})))^{a_1}$, and $m_4 \leq 3e^{2a_0}E(e^{2\delta_t})(E(e^{2\Delta_{t-1}}))^{a_1}$. Due to stationarity we get $\frac{1}{3}m_4 = E(Y_t) = E(Y_{t-1})$ (from equation (1) above) and therefore $m_4 \leq 3e^{2a_0}E(e^{2\delta_t})(\frac{1}{3})^{a_1}m_4^{a_1}$. Solving with respect to m_4 we get $m_4^{1-a_1} \leq 3^{1-a_1}e^{2a_0}E(e^{2\delta_t})$. Since $\delta_t \in N(0,1)$ we have

$$\begin{aligned} E(e^{2\delta_t}) &= \int_{\mathbb{R}} e^{2x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-4x+4-4)}dx \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2+2}dx \qquad \{u = x-2\} \\ &= e^2\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du \\ &= e^2 \end{aligned}$$

and therefore

$$m_4 \leq (3^{1-a_1}e^{2a_0}e^2)^{1/(1-a_1)} \leq 3e^{2(a_0+1)/(1-a_1)}$$

Finally $a_0 > 0$ and $0 < a_1 < 1$ so $a_0 + 1 > 1$ and $1 - a_1 \leq 1$ and therefore

$$E(h_t^4) = m_4 \leq 3e^2$$

□